A New Class of $\mathcal{N} = 2$ Topological Amplitudes

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work in collaboration with I. Antoniadis (CERN), K.S. Narain (ICTP Trieste) and E. Sokatchev (LAPTH Annecy)

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A well known example of $\mathcal{N}=2$ topological amplitudes is the following equivalence between correlators of two quite different theories:

Antoniadis, Gava, Narain, Taylor, 1993

$$F_g = \langle R^2_{(+)}\, T^{2g-2}_{(+)} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}$$

g-loop correlator in type II string theory on CY_3 (insertions from $\mathcal{N}=2$ SUGRA multiplet)

genus g partition function of the $\mathcal{N}=2$ (closed) topological string

Introduction: $\mathcal{N} = 2$ Topological Amplitudes

Some more details

$$F_g = \langle R^2_{(+)} T^{2g-2}_{(+)} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}$$

The corresponding effective action terms on the string side can be written in a manifestly $\mathcal{N} = (2,2)$ supersymmetric manner Antoniadis, Gava, Narain, Taylor, 1993

$$S = \int d^4x \int d^4\theta (\epsilon_{ij}\epsilon_{kl} \mathcal{W}^{ij}_{\mu\nu} \mathcal{W}^{kl}_{\mu\nu})^g F_g(X^I)$$

with the Weyl multiplet

$$\mathcal{W}^{ij}_{\mu\nu} = T^{ij}_{(+),\mu\nu} - (\theta^i \sigma^{\lambda\rho} \theta^j) R_{(+),\mu\nu\rho\tau}$$

- To be compatible with the superspace measure, $F_g(X^I)$ can only depend on the chiral vector multiplets X^I (holomorphicity condition)
- These couplings are exact to all orders receiving neither additional higher order nor non-perturbative corrections.

Introduction: $\mathcal{N} = 2$ Topological Amplitudes

Some more details

$$F_g = \langle R_{(+)}^2 T_{(+)}^{2g-2} \rangle_{g-\text{loop}} = \int_{\mathcal{M}_g} \langle \prod_{a=1}^{3g-3} |G^-(\mu_a)|^2 \rangle_{\text{top}}$$

To understand the G^- on the topological side we start with an $\mathcal{N} = (2,2)$ SCFT spanned by the operators

$$\{T, G^\pm, J | \overline{T}, \overline{G}^\pm, \overline{J}\}$$

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The twist is performed in the following manner

Witten, 1992
Bershadsky, Cecotti, Ooguri, Vafa, 1993
Cecotti, Vafa, 1993

$$T \to T - \frac{1}{2}\partial J, \qquad \overline{T} \to \overline{T} - \frac{1}{2}\overline{\partial J}$$

In this way G^- acquires conformal dimension 2 and can be sewed with the Beltrami differentials μ_a to form the topological integral measure.

Besides being interesting in their own right, topological amplitudes (in general) have attracted a lot of interest in many instances

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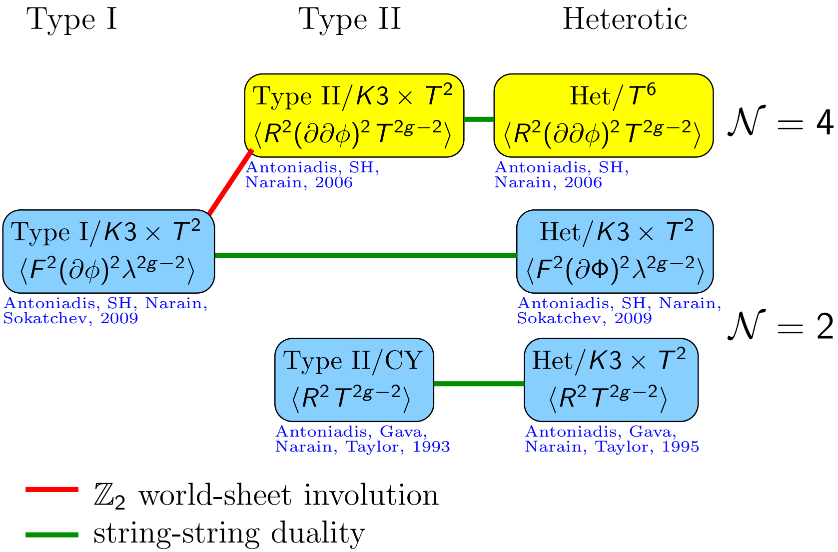
These are also good reasons to find new classes of topological amplitudes!


Type I
Type II
Heterotic
Type II/$K3 \times T^2$
$\langle R^2(\partial\partial\phi)^2 T^{2g-2}\rangle$
Antoniadis, SH, Narain, 2006
Het/T^6
$\langle R^2(\partial\partial\phi)^2 T^{2g-2}\rangle$
Antoniadis, SH, Narain, 2006
$\mathcal{N} = 4$
Type I/$K3 \times T^2$
$\langle F^2(\partial\phi)^2 \lambda^{2g-2}\rangle$
Antoniadis, SH, Narain, Sokatchev, 2009
Het/$K3 \times T^2$
$\langle F^2(\partial\Phi)^2 \lambda^{2g-2}\rangle$
Antoniadis, SH, Narain, Sokatchev, 2009
$\mathcal{N} = 2$
Type II/CY
$\langle R^2 T^{2g-2}\rangle$
Antoniadis, Gava, Narain, Taylor, 1993
Het/$K3 \times T^2$
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$\mathbb{Z}_2$ world-sheet involution
string-string duality

Outline of the Remainder of the Seminar

1. New Topological Amplitudes in String Theory
   - New Topological Amplitudes in Heterotic String Theory/$K3 \times T^2$
   - Manifestly Supersymmetric Effective Action Couplings
2. Differential Equations
   - Holomorphicity Relation
   - Harmonicity Relation and Second Order Constraint

New Topological Amplitudes in Heterotic/$K3 \times T^2$

$$\mathcal{F}_g^{(2)} = \langle F_{(+)}^2 (\partial\Phi)^2 (\lambda_\alpha \lambda^\alpha)^{g-2} \rangle_g^{\text{het}} =$$

$$= \int_{\mathcal{M}_g} \langle \prod_{a=1}^{g} G_{T^2}^-(\mu_a) \prod_{b=g+1}^{3g-4} G_{K3}^-(\mu_b) J_{K3}^{--}(\mu_{3g-3}) \psi_3 (\det Q_i)(\det Q_j) \rangle$$

Antoniadis, SH, Narain, Sokatchev, 2009

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The world-sheet theory on $K3 \times T^2$ is a product theory

$$\{T_{T^2}, G_{T^2}^{\pm}, J_{T^2}\} \times \{T_{K3}, G_{K3}^{\pm}, \tilde{G}_{K3}^{\pm}, J_{K3}, J_{K3}^{\pm\pm}\}$$

Banks, Dixon 1988
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Twisting of this theory is done by picking an $\mathcal{N} = 2$ subalgebra

$$T_{T^2} + T_{K3} \rightarrow T_{T^2} + T_{K3} - \frac{1}{2}\partial(J_{T^2} + J_{K3}),$$

This is a semi-topological correlator (twisting only in the SUSY sector)

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Antoniadis, SH, Narain, Sokatchev, 2009

- ψ_3 is a free fermion on the torus (necessary to soak zero modes)
- Q_i are the zero modes of the right moving (bosonic) currents in the heterotic theory

$$\mathcal{F}_g^{(2)} = \langle F_{(+)}^2 (\partial\Phi)^2 (\lambda_\alpha \lambda^\alpha)^{g-2} \rangle_g^{\text{het}} =$$

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Antoniadis, SH, Narain, Sokatchev, 2009

- g-loop amplitude in heterotic string theory on $K3 \times T^2$
- Component correlator with insertions from $\mathcal{N} = 2$ vector multiplet:
  - $F_{(+),\mu\nu}$ gauge field strength
  - Φ vector multiplet scalars
  - λ_α gaugino
- Supersymmetrization involves hypermultiplets

Manifest Supersymmetric Effective Action Couplings

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$$\mathbb{R}^{(4+4|2,2)} = \mathbb{R}^{(4|2,2)} \times \frac{SU(2)}{U(1)} = \{x^\mu, \theta^\pm_\alpha, \bar{\theta}^{\dot{\alpha}}_\pm, u^\pm_i\}$$

with the harmonic variables

$$\frac{SU(2)}{U(1)} = \{u_i^+,\, u_i^-\} \qquad \text{with} \qquad \begin{cases} i = 1,2 & \in & SU(2) \\ \pm & \dots & U(1) \end{cases}$$

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The Grassmann variables are $SU(2)$-projected

$$\theta^\pm_\alpha = \theta^i_\alpha\, u^\pm_i, \qquad \text{and} \qquad \bar{\theta}^{\dot{\alpha}}_\pm = \bar{\theta}^{\dot{\alpha}}_i \bar{u}^i_\pm$$

leading to the measure on the harmonic superspace

$$\int d\zeta^{(-2,-2)} = \int\, d^4x\; du\; d^2\theta^+ d^2\bar{\theta}_-\,,$$

Hypermultiplets in Harmonic Superspace

We first introduce N doublets of hypermultiplets transforming as fundamentals under $SO(N)$

$$q^+_{\hat{A}} = f^+_{\hat{A}} + \theta^+_\alpha \, \chi^\alpha_{\hat{A}} + \bar{\psi}_{\hat{A}\dot{\alpha}} \, \bar{\theta}^{\dot{\alpha}}_- + \ldots$$

$$\tilde{q}_{\hat{A}-} = \bar{f}_{\hat{A}-} + \bar{\theta}^{\dot{\alpha}}_- \, \bar{\chi}_{\hat{A}\dot{\alpha}} + \psi^\alpha_{\hat{A}} \, \theta^+_\alpha + \ldots$$

with $\hat{A} \in SO(N)$

These can be combined into $SU(2)$ doublets in the following manner

$$(q^+_{\hat{A}}, \tilde{q}_{\hat{A}-}) = q^+_{\hat{A}a} = q^+_A \,, \qquad \begin{cases} a \in SU(2) \\ A \in Sp(2N) \end{cases}$$

These superfields satisfy particular analyticity relations

$$D^\alpha_- q^+_A = \bar{D}^+_{\dot{\alpha}} q^+_A = 0$$

The vector multiplets have the expansion

$$W_I = \varphi_I + \theta^i_\alpha \lambda^\alpha_{iI} + \theta^i_\alpha \theta^j_\beta \left(\epsilon_{ij} F^{(\alpha\beta)}_{(+),I} + \epsilon^{\alpha\beta} S_{(ij),I} \right)$$

φ_I → scalar, λ^α_{iI} → gauginos, $F^{(\alpha\beta)}_{(+),I}$ → gauge field, $S_{(ij),I}$ → auxiliary

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$$\swarrow \qquad \downarrow \qquad \downarrow \qquad \searrow$$

scalar   gauginos   gauge field   auxiliary

We will also consider the superdescendant

$$K^\alpha_{-,I} = \bar{u}^i_- D^\alpha_i \; W_I = \lambda^\alpha_{iI} \bar{u}^i_- + i(\sigma^\mu)^{\alpha\dot{\alpha}} \bar{\theta}^+_{\dot{\alpha}} \partial_\mu \varphi_I + \theta^+_\beta \; F^{\alpha\beta}_{(+),I}$$

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On shell (for $S_{(ij)} = 0$), both superfields satisfy analyticity conditions

$$\epsilon_{\alpha\beta} D^\alpha_i D^\beta_j W_I = 0 \qquad \text{and} \qquad D^\beta_- K^\alpha_{-,I} = \bar{D}^+_{\dot{\alpha}} K^\alpha_{-,I} = 0$$

In the following we will mostly suppress the vector index I.

Higher-Derivative Couplings

The coupling corresponding to the topological amplitude is then given by

Antoniadis, SH, Narain, Sokatchev, 2009

$$S_2 = \int d\zeta^{(-2,-2)}(D_-^\alpha \epsilon_{\alpha\beta} D_-^\beta)\left[(K_-^\alpha \epsilon_{\alpha\beta} K_-^\beta)^{g-1}\,\tilde{\mathcal{F}}_g^{(2)}(W, q_A^+, u)\right]$$

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This term is (off-shell) supersymmetric since it is annihilated by $(D^\pm, \bar{D}_\pm)$

- Acting with D^α_+ and $\bar{D}^-_{\dot\alpha}$ vanishes due to the measure factor
- Acting with D^α_- vanishes because of the presence of $(D^\alpha_- \epsilon_{\alpha\beta} D^\beta_-)$
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Notice, that the coupling function $\tilde{\mathcal{F}}^{(2)}_g(W, q^+_A, u)$ does not depend on the superfields in an arbitrary way but satisfies certain analyticity properties.

Analyticity Properties of the Topological Amplitudes

To see these properties more clearly let us write the amplitude in an on-shell formulation ($S_{(ij)} = 0$)

$$\int d\zeta^{(-2,-2)}(K_-^\alpha \epsilon_{\alpha\beta} K_-^\beta)^g \mathcal{F}_g^{(2)}(W_I, q_A^+, u)$$

- It is crucial to notice that $\mathcal{F}_g^{(2)}$ does not depend on the moduli in a random way
- Particularly, it just depends on
  - the holomorphic vector multiplets
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- These analyticities suggest differential equations for $\mathcal{F}_g^{(2)}$
  - holomorphic anomaly equation with respect to $\bar{\varphi}^{\bar{I}}$
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- Particularly, it just depends on
  - the holomorphic vector multiplets
  - a particular projection of the hypermultiples q_A^+
- These analyticities suggest differential equations for $\mathcal{F}_g^{(2)}$
  - holomorphic anomaly equation with respect to $\bar{\varphi}$
  - harmonicity relation and second order equation for the hyper multiplets

Holomorphicity Relation

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$$\frac{\partial}{\partial\bar{\varphi}^{\bar{I}}}\mathcal{F}_g^{(2)} = 0$$

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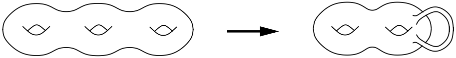
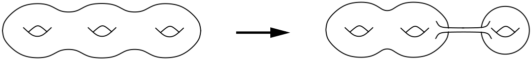
In general there are two types of degenerations

- Degeneration of a handle



- Degeneration of a dividing geodesic



Holomorphic Anomaly

For computing the violation of the holomorphicity, we need to consider the states propagating on the thin long tubes

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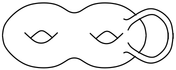

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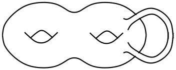
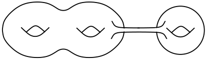


- Pinching a dividing geodesic
  Uncharged vector multiplet states can contribute. Due to the necessity to soak up torus zero-modes the contribution vanishes unless one of the two surfaces happens to be a torus $\Rightarrow$ Only contribution for $g \to (g-1)+1$



Holomorphic Anomaly

The contribution of the torus can be calculated explicitly yielding the result

$$\frac{\partial}{\partial \bar{\varphi}^{\bar{I}}} \mathcal{F}_g^{(2)} = \mathcal{F}_{\bar{I},\bar{K}}^{g-1,1} G^{\bar{K}L} \partial_L h^{(1)}$$

Antoniadis, SH, Narain, Sokatchev, 2009

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- $h^{(1)}$ is the one-loop threshold correction to the gauge-couplings
- $\mathcal{F}_{\bar{I},\bar{K}}^{g-1,1}$ is a new topological object. It is a non-holomorphic coupling in the effective action

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- $\mathcal{F}_{\bar{I},\bar{K}}^{g-1,1}$ is a new topological object. It is a non-holomorphic coupling in the effective action
- The superspace couplings for $\mathcal{F}_{\bar{I},\bar{K}}^{g-1,1}$ can be interpreted as an anomaly to the holomorphicity condition, generalizing the well-known holomorphic anomaly equation. Bershadsky, Cecotti, Ooguri, Vafa, 1993

Holomorphic Anomaly

The contribution of the torus can be calculated explicitly yielding the result

$$\frac{\partial}{\partial \bar{\varphi}^{\bar{I}}} \mathcal{F}_g^{(2)} = \mathcal{F}_{\bar{I},\bar{K}}^{g-1,1} G^{\bar{K}L} \partial_L h^{(1)}$$

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- The phenomenon of the holomorphicity relation not closing on $\mathcal{F}_g^{(2)}$ is not new. A similar observation was already made for semi-topological $\mathcal{N} = 1$ amplitudes in the heterotic theory compactified on CY

  Antoniadis, Gava, Narain, Taylor, 1996

Harmonic Dependence of the Topological Amplitudes

Let us consider the harmonic dependence of $\mathcal{F}_g^{(2)}$ by the generic expansion (I drop the W-dependence, $m = 2g - 2$)

$$\mathcal{F}_g^{(2)}(q_A^+, u) = \sum_{n=0}^{\infty} \xi^{A_1 \ldots A_n}_{(i_1 \ldots i_{m+n})} \bar{u}_+^{i_1} \ldots \bar{u}_+^{i_{m+n}} f_{A_1}^{(k_1} \ldots f_{A_n}^{k_n)} u_{k_1}^+ \ldots u_{k_n}^+ =$$

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- harmonicity relation

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$$\epsilon^{ij} D_{i,A} D_{j,B} \mathcal{F}_g^{(2)} = 0.$$

Anomalies for the Harmonicity Relation

Also the harmonicity relation is modified by boundary corrections similar to the holomorphic anomaly equation. Explicit string computations at a generic point in the moduli space show Antoniadis, SH, Narain, Sokatchev 2009

$$\epsilon^{ij}\frac{\partial}{\partial\bar{u}^i_+}D_{j,A}\mathcal{F}^{(2)}_g=\sum_{g_1=2}^{g-2}D_{A+}D_{B+}\mathcal{F}^{(2)}_{g_1}\Omega^{BC}D_{C+}\mathcal{F}^{(2)}_{g-g_1}+$$
$$+\mathcal{F}^{(2)}_{1,AB}\Omega^{BC}D_{C+}\mathcal{F}^{(2)}_{g-1}+$$
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Here $\mathcal{F}^{g-1,1}_{A,\bar{K}}$ is again a new non-holomorphic coupling in the effective action, which contributes to this amplitude via the elimination of the auxiliary fields $S_{(ij)}$. Ω^{AB} is the symplectic form of $Sp(2N)$.

Anomalies for the Second Order Relation

Finally, also the second order relation is modified. Besides the usual boundary contributions we find Antoniadis, SH, Narain, Sokatchev 2009

$$\epsilon^{ij}\, D_{i,\hat{A}a} D_{j,\hat{B}b} \mathcal{F}_g^{(2)} = (g-1)\delta_{\hat{A}\hat{B}}\epsilon_{ab}\mathcal{F}_g^{(2)} + \text{boundary terms}$$

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- It plays the role of a connection term owing to the fact that the space of the f_i is not flat.
- The presence of this term can also be understood from the field theoretic/superspace point of view.

Conclusions

In this talk I have presented a new class of $\mathcal{N} = 2$ topological amplitudes.

- I determined the corresponding effective action couplings in harmonic superspace and found that these topological couplings depend on both vector- and hypermultiplet moduli
- I showed that these couplings satisfy certain differential equations with respect to the moduli, namely
  - holomorphicity relation with respect to vector moduli
  - harmonicity and second-order relation with respect to hyper moduli
- Open questions still include
  - What do these amplitudes compute mathematically?
  - Are there any physical applications?