Phase transitions in large N symmetric orbifold CFTs

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Motivation/Overview

We want to investigate phase transitions of large N CFTs in two dimensions

Two different point of views:

- as a purely field theoretic exercise: $\Rightarrow$ Hagedorn transitions
- as duals to AdS_3: $\Rightarrow$ Hawking-Page transitions

The methods we use are quite different from $d \geq 3$ $SU(N)$ gauge theories

Our main tool is *modular invariance*.

Main motivation

We want to be able to investigate non-holomorphic theories.

The field theoretic view: partition function

Consider a 2D conformal field theory with left- and right-moving central charges such that $c_L - c_R = 0\,(mod\,24)$. Moreover, we assume that the theory has a unique vacuum and a gap,*i.e.* $h_\phi, \bar{h}_\phi \geq h$ for all states ϕ. We define the theory on a circle whose radius sets the unit of length.

The partition function takes the general form

$$Z(\tau,\bar{\tau}) = \mathrm{Tr}\, q^{L_0 - c_L/24}\bar{q}^{\bar{L}_0 - c_R/24} = q^{-c_L/24}\bar{q}^{-c_R/24}\sum_{m,\bar{m}} \tilde{d}(m,\bar{m})q^m\bar{q}^{\bar{m}}\ ,$$

where $\tilde{d}(m,\bar{m})$ is the number of states at left- and right-moving levels $(m,\bar{m})$, and

$$q = e^{2\pi i\tau} = e^{-\beta + i\mu}\ .$$

Here β is the inverse temperature and μ is the variable conjugate to the momentum on the circle.

Often, we will set $\mu = 0$ and focus on the behavior of Z as a function of β.

Modular invariance: $SL(2,\mathbb{Z})$

The main advantage of the 2d case is that $Z(\tau,\bar{\tau})$ is *modular invariant*:

We can define amplitudes of the CFT on the torus. These amplitudes can only depend on the conformal structure of the torus, τ.

The torus remains unchanged if we transform τ under the modular group $SL(2,\mathbb{Z})$:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

The 0-point amplitude $Z(\tau,\bar{\tau})$ must therefore also remain unchanged under modular transformations.

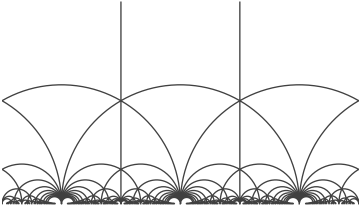
The phase diagram

Modular invariance allows us to reconstruct the phase space diagram.

The free energy F is given by

$$Z(\tau, \bar{\tau}) = e^{2\pi i \tau F(\tau, \bar{\tau})}$$

If we know $F(\tau, \bar{\tau})$ in one fundamental region of $SL(2, \mathbb{Z})$, we know it everywhere. Standard fundamental region: $|\tau| \geq 1, |\Re(\tau)| \leq \frac{1}{2}$
A tesselation of the upper half plane into fundamental regions:



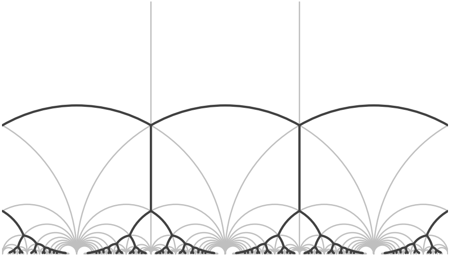
(taken from [Witten,Maloney])

The phase diagram II

It is natural to conjecture that the phase diagram will correspond to this picture. For symmetric orbifolds, this is indeed almost what happens.

In fact F is invariant under $\tau \mapsto \tau = \tau + 1$.

The expected phase diagram therefore looks more like



A more detailed analysis

We now want to analyse in more detail when and where phase transitions occur.

For finite c_L, c_R there are only finitely many degrees of freedom, so there cannot be a phase transition (on a finite volume.)

Phase transitions can only occur in the limit where the central charges go to infinity.

$\Rightarrow$ we shall consider families of two dimensional CFTs $\{\mathcal{C}^{(N)}\}_{N\in\mathbb{N}}$ whose central charges are given by $c_{L,R}(N) = c_{L,R}N$

Central questions:

- What families of CFTs have phase transitions in the large N limit?
- Where do these phase transitions occur?

A simple example: Hagedorn transitions

For the moment assume $\mu = 0$, *i.e.* we concentrate on the imaginary axis of the phase diagram.

$$Z(\beta) = e^{\beta(c_L+c_R)/24} \sum_{m,\bar{m}} \tilde{d}(m,\bar{m}) e^{-\beta(m+\bar{m})}$$

For a Hagedorn transition (*i.e.* F divergent at finite β_0) the number of states should grow exponentially,

$$\tilde{d}(m,\bar{m}) \sim e^{\beta_0 m} e^{\beta_0 \bar{m}}$$

However, from the Cardy formula we know that

$$d(m) \sim e^{2\pi\sqrt{c_L m/6}} ,$$

so that no Hagedorn tranistion occurs. (Alternatively, this follows directly from modular invariance.)

Validity of the Cardy formula

$$Z(q) = \sum_n d(n) q^n$$

Modular invariant partition function. Use the $SL(2,\mathbb{Z})$ transformation $\tau \mapsto -1/\tau$ to obtain the high temperature behavior from the low temperature behavior:

$$d(n) = \oint_C dq q^{-n-1} Z(\tau) = \oint_C dq q^{-n-1} Z(-1/\tau) \, ,$$

Use saddle point approximation to obtain

$$d(n) \sim \left(\frac{c}{96n^3}\right)^{1/4} e^{2\pi\sqrt{cn/6}} \, .$$

Usually, this is only valid for $n \gg c$. If the partition function has a parametrically large gap L,

$$Z(q) = q^{-c/24}(1 + q^L + \ldots) \, ,$$

then it already holds for $n > c/24$.

Universality of symmetric orbifold theories

To put it another way, for an arbitrary CFT

- the low temperature behavior of the free energy is universal
- the high temperature behavior is universal (following from modular invariance)

$$\log Z \simeq \begin{cases} \frac{\pi(c_L+c_R)}{12}\,\frac{\beta}{2\pi} & \text{for } \beta \gg 1\,, \\ \frac{\pi(c_L+c_R)}{12}\,\frac{2\pi}{\beta} & \text{for } \beta \ll 1\,. \end{cases}$$

- the behavior in between however depends on the details of the theory

For symmetric orbifold theories we will show that in the large N limit, the free energy is universal.

$\Rightarrow$ In this case the low and high temperature regime extend all the way to medium temperature.

Modular invariance on the gravity side

How can we understand the modular properties of the partition function from the AdS_3 gravity point of view?

The classical solutions of Euclidean solutions of Einstein's equations with negative cosmological constants are essentially given by [Maldacena, Strominger]

$$AdS_3/\Gamma \; , \qquad \Gamma \in SL(2,\mathbb{Z})/\mathbb{Z}_2$$

More precisely, for a given conformal boundary there is a family of classical solutions

$$\mathcal{M}_{c,d} \; : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

i.e. there is exactly one solution for every element of the modular group (up to translations T).

Summing over all solutions gives something modular invariant.

Poincaré sums

Mathematically, this can be made more precise. The sum over all images of the modular group is called a *Poincaré sum*.

The main problem is that these sums are divergent and need to be regularized.

For holomorphic objects, it is relatively clear how this works:

- holomorphic extremal/chiral theories [Manschot], [Maloney, Song, Strominger]
- elliptic genus of $N = 2$ supersymmetric theories [Dijkgraaf, Maldacena, Moore, Verlinde] [Manschot, Moore]

The main reason this works so well is that for holomorphic modular objects, a finite set of terms ('polar terms') fixes the entire series.

A toy model

To illustrate how phase transitions come about, let us consider the classical contribution of holomorphic/chiral theory with $c = 24N$,

$$f(q) = q^{-N}$$

The (unregularized) sum over images is

$$f_N(q) = \sum_{(c,d)=1} e^{-2\pi i N \frac{a\tau+b}{c\tau+d}}$$

For τ in a given fundamental region, there is exactly one term that is dominant. In the $N \to \infty$ limit, the contribution of the other terms can be neglected. $\Rightarrow f_\infty$ is non-analytic on the boundaries of the fundamental regions

Problem: for a general $f(q)$,

$$f(q) = q^{-N} + \tilde{d}_N(1) q^{-N+1} + \dots$$

the subleading terms can smooth out the discontinuity

$\Rightarrow$ need to control these contributions

The setup: large N families

Consider families $\{\mathcal{C}^{(N)}\}_{N\in\mathbb{N}}$ of CFTs whose central charges are given by $c_{L,R}(N) = c_{L,R}N$. From the asymptotic behaviors we expect the free energy to scale linearly with N. We will therefore define the following limit:

$$-\beta f(\beta) \;\equiv\; \lim_{N\to\infty} \frac{1}{N} \log Z_N$$

Main question:

Is $f(\beta)$ analytic, or does it have phase transitions?

To put it another way, each such family will have a Hagedorn transition in the $N \to \infty$ limit (since the free energy diverges.) After rescaling with N however the resulting function could still be analytic.

From the gravity point of view this rescaling is natural: The energy scale is given by the AdS radius, which is proportional to N.

Tensor products

Let us briefly return to our toy model. Since we take τ purely imaginary, it sufficient to keep only two terms of the Poincaré sum,

$$Z(q) = e^{\beta N \frac{c_L + c_R}{24}} (1 + d_N(h_1) e^{-h_1 \beta} + \ldots) \simeq e^{\beta N \frac{c_L + c_R}{24}} + e^{4\pi^2 N \frac{c_L + c_R}{24} / \beta}$$

since for N large the other states don't contribute.

This gives

$$f(\beta) = \begin{cases} -\frac{c_L + c_R}{24} + o(1) & : \beta > 2\pi \\ -\frac{4\pi^2}{\beta^2} \frac{c_L + c_R}{24} + o(1) & : \beta < 2\pi \end{cases}$$

$\Rightarrow$ there is a phase transition at $\beta = 2\pi$.

However, this approximation cannot be valid in general. For N copies of a given CFT tensored together, $f(\beta) = f_0(\beta)$, so that there is no phase transition.

Symmetric orbifolds

For the tensor product of CFTs, the number of low-lying states $d_N(h_n)$ obviously grows quickly with N, which is enough to smooth out the behavior of $f(\beta)$.

We introduce an orbifold to eliminate these states.

Symmetric orbifold

$$CFT_N = \underbrace{CFT_1 \otimes \cdots \otimes CFT_1}_{N} / S_N$$

- project out invariant states
- introduce twisted sectors

Note: the 'seed theory' CFT_1 has a gap and satisfies $c_L - c_R = 0\,(mod\,24)$. Other than that, it is completely arbitrary.

Symmetric orbifold partition functions

There are closed expressions for the generating function $\mathcal{Z}$ of the partition functions Z_N of symmetric orbifold CFTs [Dijkgraaf, Moore, Verlinde,Verlinde]

$$\mathcal{Z} = \sum_{N\geq 0} p^N Z(S^N CFT_1) = \prod_{n>0} \prod_{m,\bar{m}\in I} (1 - p^n q^{m/n} \bar{q}^{\bar{m}/n})^{-d(m,\bar{m})\delta^{(n)}_{m-\bar{m}}}$$

Alternatively, to make the modular properties of the partition function manifest, we can also express $\mathcal{Z}$ in terms of Hecke operators,

$$\mathcal{Z} = \exp\left(\sum_{L>0} \frac{p^L}{L} T'_L Z(q,\bar{q}) \right) ,$$

where the action of the Hecke operators is defined by

$$T'_L f(\tau,\bar{\tau}) = \sum_{d|L} \sum_{b=0}^{d-1} f\left(\frac{L\tau + bd}{d^2}, \frac{L\bar{\tau} + bd}{d^2} \right) .$$

Hecke operators map modular invariant objects to modular invariant objects.

The $N \to \infty$ limit

Let us shift the vacuum energy to zero to define

$$\tilde{Z}_N(\tau, \bar{\tau}) = q^{c_L L/24} \bar{q}^{c_R L/24} Z_N(\tau, \bar{\tau})$$

We are thus interested in the object $\tilde{Z}_\infty(\tau, \bar{\tau})$. It will turn out to be completely universal, independent of the details of the 'seed theory' CFT_1.

It turns out that

$$\tilde{Z}_\infty(\tau, \bar{\tau}) = \sum_{n, \bar{n}} \tilde{d}_\infty(n, \bar{n}) q^n q^{\bar{n}}$$

is actually well-defined. This follows from the fact that the number of states $d_N(n, \bar{n})$ is independent of N for $n, \bar{n} < n_0 N$ (where n_0 is some constant related to the gap of the seed theory.)

In fact, the leading behavior of the number of states is given by

$$\tilde{d}_\infty(n, \bar{n}) \sim e^{2\pi n} e^{2\pi \bar{n}}$$

Note that this is consistent with the fact that there is a Hagedorn transition for $\beta_H = 2\pi$.

The large N behavior

We will now derive a formula for the large N behavior. For convenience we shift the vacuum energy to zero by introducing $\tilde{p} = pq^{-\frac{c_L}{24}}\bar{q}^{-\frac{c_R}{24}}$ Defining the action of T'_L by

$$T'_L Z(q,\bar{q}) = q^{-c_L L/24}\bar{q}^{-c_R L/24}\left(1 + \sum_{n,\bar{n}>0} \tilde{d}_L(n,\bar{n}) q^n \bar{q}^{\bar{n}}\right)$$

we can write

$$\mathcal{Z} = \exp\left(\sum_{L>0}\frac{1}{L}\tilde{p}^L + \sum_{L>0}\frac{1}{L}\tilde{p}^L \sum_{n,\bar{n}\geq n_0 L} \tilde{d}_L(n,\bar{n}) q^n \bar{q}^{\bar{n}}\right)$$

$$= \left(\sum_{K\geq 0}\tilde{p}^K\right)\exp\left(\sum_{L>0}\frac{1}{L}\tilde{p}^L \sum_{n,\bar{n}\geq n_0 L} \tilde{d}_L(n,\bar{n}) q^n \bar{q}^{\bar{n}}\right)$$

$\tilde{Z}_N$ is then simply the coefficient of $\tilde{p}^N$.

The large N behavior II

Let us assume for the moment that $\beta < 2\pi$. To evaluate $\tilde{d}(n,\bar{n})$ we use again the Cardy formula given in the previous section. We then evaluate the sum over $n,\bar{n}$ by using a saddle point approximation,

$$\int dn \left(\frac{c_L L}{96(n - \frac{c_L L}{24})^3} \right)^{1/4} e^{2\pi\sqrt{\frac{c_L}{6}L(n-\frac{c_L}{24})}} e^{-\beta n} = 1 \cdot e^{(\frac{4\pi^2}{\beta}-\beta)\frac{c_L L}{24}} + \text{subleading} \; .$$

Note that the use of the Cardy formula is permissible for large L since the saddle point is at $n_0 = (4\pi^2\beta^{-2}+1)cL/24 + O(1)$, and the modular form has a parametrically large gap.

Also note the contribution grows exponentially in L, so that we can neglect the contribution of small L.

We can then perform the sum over L and read off the coefficient of $\tilde{p}^N$,

$$\tilde{Z}_N(\beta) = \sum_{L=0}^{N} e^{L(\frac{4\pi^2}{\beta}-\beta)(c_L+c_R)/24} \; , \qquad \text{for } \beta < 2\pi \; .$$

The large N behavior, III

We can then use modular invariance of Z_N to obtain the expression for $\tilde{Z}_N$ in the low temperature regime as well. It turns out to be given by the same expression,

$$\tilde{Z}_N(\beta) = \frac{e^{(N+1)(\frac{4\pi^2}{\beta}-\beta)(c_L+c_R)/24} - 1}{e^{(\frac{4\pi^2}{\beta}-\beta)(c_L+c_R)/24} - 1} , \qquad \text{for } \left|4\pi^2\beta^{-1} - \beta\right| > O(N^{-1}) .$$

Going to the $N \to \infty$ limit, this expression gives the expected result

$$f(\beta) = \begin{cases} -\frac{c_L+c_R}{24} & : \beta > 2\pi \\ -\frac{4\pi^2}{\beta^2}\frac{c_L+c_R}{24} & : \beta < 2\pi \end{cases}$$

This shows that the free energy in the large N limit is completely universal.

The full phase space diagram

So far we have always set $\mu = 0$, so that only one phase transition (at $\beta = 2\pi$) occurs.

One can show that the full diagram has the following form: In the standard fundamental region $\mathcal{F}$, *i.e.* for $|\tau| > 1, |\Re(\tau)| \leq \frac{1}{2}$ the free energy f_N is given by

$$f_N(\tau) = -\frac{(c_L + c_R)}{24} + O(N^{-1}) \ : \qquad \tau \in \mathcal{F} \, .$$

For all other values of τ the value of f_N can be obtained by applying a suitable modular transformation.

This shows that the phase diagram is really given by the tesselation into fundamental regions of $SL(2, \mathbb{Z})/T$.

Sketch of the proof

We know that $\tilde{Z}_\infty$ actually exists, so we can try to extract it directly. For this it is useful to use the expression

$$\mathcal{Z} = (1-\tilde{p})^{-1} \prod_{n>0,m,\bar{m}\in I}' (1-\tilde{p}^n q^{m/n+nc_L/24}\bar{q}^{\bar{m}/n+nc_R/24})^{-d(m,\bar{m})\delta^{(n)}_{m-\bar{m}}}$$
$$= (1-\tilde{p})^{-1} R(\tilde{p}) \,,$$

where we have assumed that $d(-c_L/24, -c_R/24) = 1$ and the primed product omits the factor with $(n = 1, m = -c_L/24, \bar{m} = -c_R/24)$.

Using the same trick as [de Boer], if R has the expansion $R(\tilde{p}) = \sum_k a_k \tilde{p}^k$, then it follows that $\tilde{Z}_N = \sum_{k=0}^N a_k$. It follows that

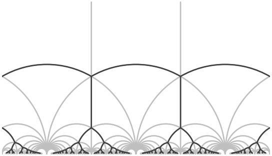
$$\tilde{Z}_\infty = \sum_{k=0}^{\infty} a_k = R(1) \,.$$

Sketch of the proof II

$$\beta^{-1}\tilde{F}_\infty = \log \tilde{Z}_\infty = -\sum_{n>0}\sum_{m,\bar{m}}{}' d(m,\bar{m})\delta^{(n)}_{m-\bar{m}} \log(1-q^{m/n+nc_L/24}\bar{q}^{\bar{m}/n+nc_R/24})$$

$$= \sum_{n>0}\sum_{m,\bar{m}}{}' d(m,\bar{m})\delta^{(n)}_{m-\bar{m}} \sum_{a=1}^{\infty} a^{-1} q^{a(m/n+nc_L/24)}\bar{q}^{a(\bar{m}/n+nc_R/24)}).$$

It is then sufficient to show that this expression converges for $\beta > 2\pi$. Using the Cardy formula for $d(m,\bar{m})$ and performing a saddle point approximation, one can show that it does converge in the fundamental region $\mathcal{F}$.



General large N families (interactions)

We have shown that symmetric orbifold theories give the phase transition behavior expected for CFT duals to AdS_3 gravity.

String theories on $AdS_3 \times \ldots$ are expected to be dual to theories in the same moduli space as symmetric orbifold CFTs, but not on the exact same spot.

To find other interesting dual CFTs, we thus need to move around in the moduli space. These other CFTs should still exhibit such a Hawking-Page transition, if they are to be viable duals to gravity theories.

We can write down a criterion for the number of state so that the theory exhibits the same phase transition behavior as above:

$$\lim_{N\to\infty} \frac{1}{N} \log \left(1 + \sum_{m,\bar{m}} \tilde{d}_N(m,\bar{m}) e^{-\beta(m+\bar{m})} \right) = 0.$$

Holomorphic theories

Unfortunately, this criterion is somewhat difficult to evaluate. For simplicity, let us concentrate on the case of holomorphic theories, $c_L = 24k, c_R = 0$.

In this case the partition function is determined by its polar and constant part. To put it another way, once we know the spectrum up to conformal weight $h = k$, we know the entire partition function.

Following [Witten], the partition function can be constructed explicitly: Let

$$J(q) = q^{-1} + 196884q + \dots$$

be the (modular invariant) partition function of the Monster CFT. Acting with the Hecke operator T'_n gives

$$T'_n J(q) = q^{-n} + O(q) .$$

It thus follows that we can write any holomorphic partition function as

$$Z_k(\tau) = \sum_{n=0}^{k} \tilde{d}_k(n) T'_{k-n} J(\tau) ,$$

Extremal CFTs

For the extremal CFTs proposed by [Witten], the coefficients are given by

$$\tilde{d}_k(n) = \left[\prod_{l=2}^{\infty} \frac{1}{1-q^l} \right]_{q^n} , \quad n = 0, \ldots, k \ .$$

In this case $\tilde{d}_k(n) \sim \exp(\pi\sqrt{2n/3})$.

The phase transitions of such theories were investigated in [Maloney, Witten]. Since the partition function is holomorphic, the phase transition comes from a Lee-Yang type condensation of zeros on the unit circle $|\tau| = 1$.

Almost extremal CFTs

We will allow for a much stronger growth in the number of states,

$$\tilde{d}(n) = e^{2\pi\alpha n} e^{o(n)} ,$$

with $o(n)$ some function such that $\lim_{n\to\infty} n^{-1} o(n)$. It follows immediately that necessarily $\alpha \leq 1$. On the other hand we can use the estimate

$$T'_n J(i) \leq n^2 (e^{2\pi n} + J(i)) .$$

to estimate

$$e^{-2\pi k} Z_k(i) \leq e^{-2\pi k} \sum_{n=0}^{k} \tilde{d}(n)(k-n)^2 (e^{2\pi(k-n)} + J(i)) \leq k^3 e^{o(k)} (J(i) + 1) ,$$

This is compatible with the criterion presented above, so there is a phase transition as long as $\alpha \leq 1$. Note that this sharpens a bound given in [Maloney, Witten].

Note that this fits nicely with the fact that symmetric orbifold theories have $\tilde{d}(n) \sim e^{2\pi n}$.

Conclusion

Conclusions

- we have analyzed the phase structure of large N CFTs
- our methods allow a systematic treatment also of non-holomorphic theories
- we have shown that symmetric orbifolds have a universal behavior in the large N limit
- we can give criteria for CFTs to exhibit Hagedorn/Hawking-Page transitions, and so constrain the kind of CFTs that can serve as gravity duals

Outlook

- investigate the moduli space of symmetric orbifolds
- understand the more complicated diagrams that can arise in interacting theories