SUSY and the brane: A σ-model story

Wieland Staessens

in collaboration with A. Sevrin and A. Wijns

based on: 0709.3733, 0809.3659, 0908.2756

Vrije Universiteit Brussel and The International Solvay Institutes

07 September 2009, Zürich

Outline

Prologue

Outline

Prologue

The story of the Closed String

Outline

Outline

Outline

Introducing the leading characters

Type II Superstrings in general NSNS background (G, H)

Introducing the leading characters

Type II Superstrings in general NSNS background (G, H)
$\rightarrow$ RNS Superstring formalism

Introducing the leading characters

Type II Superstrings in general NSNS background (G, H)
$\rightarrow$ RNS Superstring formalism
2 dim $\mathcal{N} = (2,2)$ SUSY non-linear σ-models

Introducing the leading characters

Type II Superstrings in general NSNS background (G, H)

$\longrightarrow$ RNS Superstring formalism

2 dim $\mathcal{N} = (2{,}2)$ SUSY non-linear σ-models

$\longrightarrow$ encode local geometry of Generalized Kähler Target manifolds

Introducing the leading characters

Type II Superstrings in general NSNS background (G, H)
$\rightarrow$ RNS Superstring formalism
2 dim $\mathcal{N} = (2,2)$ SUSY non-linear σ-models
$\rightarrow$ encode local geometry of Generalized Kähler Target manifolds

Why interesting?

- physics: Stringy description of flux compactifications without RR fluxes
- mathematics: playground for Generalized Kähler Geometry

Introducing the leading characters

Type II Superstrings in general NSNS background (G, H)
$\rightarrow$ RNS Superstring formalism
2 dim $\mathcal{N} = (2{,}2)$ SUSY non-linear σ-models
$\rightarrow$ encode local geometry of Generalized Kähler Target manifolds

Why interesting?

- physics: Stringy description of flux compactifications without RR fluxes
- mathematics: playground for Generalized Kähler Geometry

Here: Include D-branes in general NSNS background preserving half of the (world-sheet) SUSY

- physics: D-branes $\sim$ gauge d.o.f and chiral matter (intersecting)
- mathematics: D-branes $\sim$ subspaces of Generalized Kähler Geometry

Introducing the leading characters

SUSY $\rightarrow$ Superspace:

1. exhibits the relation between (extended) SUSY on world-sheet and (generalized) complex geometry on target space
2. facilitates the analysis
3. full off-shell $\mathcal{N}$ =(2,2) superspace description is known

Introducing the leading characters

SUSY $\rightarrow$ Superspace:

1. exhibits the relation between (extended) SUSY on world-sheet and (generalized) complex geometry on target space
2. facilitates the analysis
3. full off-shell $\mathcal{N}$ =(2,2) superspace description is known

Boundary SUSY $\rightarrow$ Boundary Superspace

1. fruitful for the local geometric description of a D-brane wrapping a subspace
2. appropriate setting for analysis of β-function
   $\Rightarrow$ Stability conditions for D-brane

Introducing the leading characters

SUSY $\rightarrow$ Superspace:

1. exhibits the relation between (extended) SUSY on world-sheet and (generalized) complex geometry on target space
2. facilitates the analysis
3. full off-shell $\mathcal{N}$ =(2,2) superspace description is known

Boundary SUSY $\rightarrow$ Boundary Superspace

1. fruitful for the local geometric description of a D-brane wrapping a subspace
2. appropriate setting for analysis of β-function
   $\Rightarrow$ Stability conditions for D-brane

Purpose of this talk

D-branes on Generalized Kähler Geometries using 2 dim SUSY σ-models in boundary superspace

Outline

The $\mathcal{N}=(1,1)$ SUSY non-linear σ-model

σ-model consists of mappings $X : \Sigma \to \mathcal{M}$

The $\mathcal{N}=$ (1,1) SUSY non-linear σ-model

σ-model consists of mappings $X : \Sigma \to \mathcal{M}$

Σ: (super)world-sheet with coordinates

- (bosonic) light-cone coordinates: $\sigma^{\ddagger} \equiv \tau + \sigma$, $\sigma^{=} \equiv \tau - \sigma$
- (fermionic) coordinates θ^{+}, θ^{-} with derivatives:
  $D_+^2 = -\frac{i}{2}\partial_{\ddagger}\ , \quad D_-^2 = -\frac{i}{2}\partial_{=}\ , \quad \{D_+, D_-\} = 0$

The $\mathcal{N}=(1,1)$ SUSY non-linear σ-model

σ-model consists of mappings $X : \Sigma \to \mathcal{M}$

Σ: (super)world-sheet with coordinates

- (bosonic) light-cone coordinates: $\sigma^{\ddagger} \equiv \tau + \sigma$, $\sigma^{=} \equiv \tau - \sigma$
- (fermionic) coordinates θ^+, θ^- with derivatives:
  $D_+^2 = -\frac{i}{2}\partial_{\ddagger}$, $\quad D_-^2 = -\frac{i}{2}\partial_{=}$, $\quad \{D_+, D_-\} = 0$

$\mathcal{M}$: target manifold with local coordinates X^a, $a \in \{1, \ldots, d\}$

- metric $g_{ab}(X)$
- Closed 3-form (torsion) $H_{abc}(X)$, locally: $H_{abc} = -\frac{3}{2}\partial_{[a}B_{bc]}$
  extra gauge symmetry $B \to B + dA$
- 2 connections $\Gamma^a_{(\pm)bc} \equiv \left\{ {a \atop bc} \right\} \pm H^a{}_{bc}$, but $\Gamma^a_{(+)bc} = \Gamma^a_{(-)cb}$

The action and extended SUSY

Using $\mathcal{N} = (1,1)$ superspace the action is simply

$$\mathcal{S}_{\mathcal{N}=(1,1)} = \int d^2\sigma d^2\theta (G_{ab} + B_{ab}) D_+ X^a D_- X^b$$

The action and extended SUSY

Using $\mathcal{N} = (1,1)$ superspace the action is simply

$$\mathcal{S}_{\mathcal{N}=(1,1)} = \int d^2\sigma d^2\theta (G_{ab} + B_{ab}) D_+ X^a D_- X^b$$

Additional world-sheet SUSY? [Alvarez-Gaumé - Freedman (1981); Gates-Hull-Roček (1984)]

$$\delta X^a = \epsilon^+ J^a_{(+)b}(X) D_+ X^b + \epsilon^- J^a_{(-)b}(X) D_- X^b$$

The action and extended SUSY

Using $\mathcal{N} = (1,1)$ superspace the action is simply

$$\mathcal{S}_{\mathcal{N}=(1,1)} = \int d^2\sigma d^2\theta (G_{ab} + B_{ab}) D_+ X^a D_- X^b$$

Additional world-sheet SUSY? [Alvarez-Gaumé - Freedman (1981); Gates-Hull-Roček (1984)]

$$\delta X^a = \epsilon^+ J^a_{(+)b}(X) D_+ X^b + \epsilon^- J^a_{(-)b}(X) D_- X^b$$

⇢ requirement 1: SUSY algebra closes (off-shell) on-shell
$\Rightarrow J_+$ and J_- are two (commuting) complex structures

The action and extended SUSY

Using $\mathcal{N} = (1,1)$ superspace the action is simply

$$\mathcal{S}_{\mathcal{N}=(1,1)} = \int d^2\sigma d^2\theta (G_{ab} + B_{ab}) D_+ X^a D_- X^b$$

Additional world-sheet SUSY? [Alvarez-Gaumé - Freedman (1981); Gates-Hull-Roček (1984)]

$$\delta X^a = \epsilon^+ J^a_{(+)b}(X) D_+ X^b + \epsilon^- J^a_{(-)b}(X) D_- X^b$$

--→ requirement 1: SUSY algebra closes (off-shell) on-shell

$\Rightarrow J_+$ and J_- are two (commuting) complex structures

--→ requirement 2: $\mathcal{N}$ =(1,1) action is invariant

$$\begin{aligned} J^T_{(\pm)} G J_\pm &= G, \\ \nabla^{(\pm)}_a J^b_{(\pm)c} &= 0 \end{aligned}$$

The action and extended SUSY

Using $\mathcal{N} = (1{,}1)$ superspace the action is simply

$$\mathcal{S}_{\mathcal{N}=(1,1)} = \int d^2\sigma d^2\theta (G_{ab} + B_{ab}) D_+ X^a D_- X^b$$

Additional world-sheet SUSY? [Alvarez-Gaumé - Freedman (1981); Gates-Hull-Roček (1984)]

$$\delta X^a = \epsilon^+ J^a_{(+)b}(X) D_+ X^b + \epsilon^- J^a_{(-)b}(X) D_- X^b$$

⇢ requirement 1: SUSY algebra closes (off-shell) on-shell
$\Rightarrow J_+$ and J_- are two (commuting) complex structures
⇢ requirement 2: $\mathcal{N}$ =(1,1) action is invariant

$$\begin{aligned} J^T_{(\pm)} G J_\pm &= G, \\ \nabla^{(\pm)}_a J^b_{(\pm)c} &= 0 \end{aligned}$$

$\Rightarrow \mathcal{M}$ characterized by $(G_{ab}, H_{abc}, J_\pm)$: Bihermitian Geometry or Generalized Kähler Geometry [Gualtieri math/0401221]

$\mathcal{N} = (2,2)$ Superspace formulation

- $\mathcal{N} = (2,2)$ superspace coordinates: $\sigma^{\ddagger}, \sigma^{=}, \theta^{+}, \theta^{-}, \hat{\theta}^{+}, \hat{\theta}^{-}$ (+ corresponding derivatives)
  $\mathcal{N} = (2,2)$ action

$$\mathcal{S}_{\mathcal{N}=(2,2)} = 4 \int d^2\sigma d^2\theta d^2\hat{\theta} V(X, \bar{X})$$

$\mathcal{N} = (2,2)$ Superspace formulation

- $\mathcal{N}= (2,2)$ superspace coordinates: $\sigma^{\ddagger}, \sigma^{=}, \theta^{+}, \theta^{-}, \hat{\theta}^{+}, \hat{\theta}^{-}$ (+ corresponding derivatives)
  $\mathcal{N} = (2,2)$ action

$$\mathcal{S}_{\mathcal{N}=(2,2)} = 4 \int d^2\sigma d^2\theta d^2\hat{\theta} V(X, \bar{X})$$

- $\mathcal{N} = (2,2)$ $\mathbb{C}$ superfields X $\rightarrow$ Too many d.o.f. and no dynamics
  $\rightarrow$ impose constraints: $\hat{D}_{\pm} X^a = J^a_{(\pm)b} D_{\pm} X^b$

$\mathcal{N}$ = (2,2) Superspace formulation

- $\mathcal{N}$= (2,2) superspace coordinates: $\sigma^{\ddagger}, \sigma^{=}, \theta^{+}, \theta^{-}, \hat{\theta}^{+}, \hat{\theta}^{-}$ (+ corresponding derivatives)
  $\mathcal{N}$ = (2,2) action

$$\mathcal{S}_{\mathcal{N}=(2,2)} = 4 \int d^2\sigma d^2\theta d^2\hat{\theta} V(X, \bar{X})$$

- $\mathcal{N}$ = (2,2) $\mathbb{C}$ superfields X $\rightarrow$ Too many d.o.f. and no dynamics
  $\rightarrow$ impose constraints: $\hat{D}_{\pm}X^a = J^a_{(\pm)b}D_{\pm}X^b$
- integrability conditions $\Rightarrow J_{(+)}, J_{(-)}$ complex structures
  3 types of superfield representation:

$\mathcal{N}$ = (2,2) Superspace formulation

- $\mathcal{N}$= (2,2) superspace coordinates: $\sigma^{\ddagger}, \sigma^{=}, \theta^{+}, \theta^{-}, \hat{\theta}^{+}, \hat{\theta}^{-}$ (+ corresponding derivatives)
  $\mathcal{N}$ = (2,2) action

$$\mathcal{S}_{\mathcal{N}=(2,2)} = 4 \int d^2\sigma d^2\theta d^2\hat{\theta} V(X, \bar{X})$$

- $\mathcal{N}$ = (2,2) $\mathbb{C}$ superfields X $\rightarrow$ Too many d.o.f. and no dynamics
  $\rightarrow$ impose constraints: $\hat{D}_{\pm} X^a = J^a_{(\pm)b} D_{\pm} X^b$
- integrability conditions $\Rightarrow$ $J_{(+)}$, $J_{(-)}$ complex structures
  3 types of superfield representation:
  1. CHIRAL $J_{(+)} = J_{(-)}$: $(z, \bar{z})$

$\mathcal{N} = (2,2)$ Superspace formulation

- $\mathcal{N}= (2,2)$ superspace coordinates: $\sigma^{\ddagger}, \sigma^{=}, \theta^{+}, \theta^{-}, \hat{\theta}^{+}, \hat{\theta}^{-}$ (+ corresponding derivatives)
  $\mathcal{N} = (2,2)$ action

$$\mathcal{S}_{\mathcal{N}=(2,2)} = 4 \int d^2\sigma d^2\theta d^2\hat{\theta} V(X, \bar{X})$$

- $\mathcal{N} = (2,2)$ $\mathbb{C}$ superfields X $\longrightarrow$ Too many d.o.f. and no dynamics
  $\longrightarrow$ impose constraints: $\hat{D}_{\pm} X^a = J^a_{(\pm)b} D_{\pm} X^b$
- integrability conditions $\Rightarrow$ $J_{(+)}$, $J_{(-)}$ complex structures
  3 types of superfield representation:
  1. CHIRAL $J_{(+)} = J_{(-)}$: $(z, \bar{z})$
  2. TWISTED CHIRAL $J_{(+)} = -J_{(-)}$: $(w, \bar{w})$

$\mathcal{N}=(2,2)$ Superspace formulation

- $\mathcal{N}=(2,2)$ superspace coordinates: $\sigma^{\ddagger},\sigma^{=},\theta^{+},\theta^{-},\hat{\theta}^{+},\hat{\theta}^{-}$ (+ corresponding derivatives)
  $\mathcal{N}=(2,2)$ action

$$\mathcal{S}_{\mathcal{N}=(2,2)}=4\int d^2\sigma d^2\theta d^2\hat{\theta}V(X,\bar{X})$$

- $\mathcal{N}=(2,2)$ $\mathbb{C}$ superfields $X\rightarrow$ Too many d.o.f. and no dynamics
  $\rightarrow$ impose constraints: $\hat{D}_{\pm}X^a=J^a_{(\pm)b}D_{\pm}X^b$
- integrability conditions $\Rightarrow J_{(+)},\ J_{(-)}$ complex structures
  3 types of superfield representation:
  1. CHIRAL $J_{(+)}=J_{(-)}$: $(z,\bar{z})$
  2. TWISTED CHIRAL $J_{(+)}=-J_{(-)}$: $(w,\bar{w})$
  3. SEMI-CHIRAL $[J_{(+)},J_{(-)}]\neq 0$: $(l,\bar{l},r,\bar{r})$

- The most general model consists of chiral, twisted chiral and semi-chiral superfields: $V(z, \bar{z}, w, \bar{w}, l, \bar{l}, r, \bar{r})$

- The most general model consists of chiral, twisted chiral and semi-chiral superfields: $V(z, \bar{z}, w, \bar{w}, l, \bar{l}, r, \bar{r})$
- Reduction $\mathcal{N} = (2,2) \rightarrow \mathcal{N} = (1,1)$ superspace by integrating out $\hat{\theta}^{+}, \hat{\theta}^{-}$ + elimination of auxiliary fields
  $\Rightarrow G_{ab}, B_{ab}, J_{(\pm)}$ as non-linear expressions of V

- The most general model consists of chiral, twisted chiral and semi-chiral superfields: $V(z, \bar{z}, w, \bar{w}, l, \bar{l}, r, \bar{r})$
- Reduction $\mathcal{N} = (2{,}2) \rightarrow \mathcal{N} = (1{,}1)$ superspace by integrating out $\hat{\theta}^{+}, \hat{\theta}^{-}$ + elimination of auxiliary fields
  $\Rightarrow G_{ab}, B_{ab}, J_{(\pm)}$ as non-linear expressions of V
- special case: chiral and/or twisted chiral $\rightarrow$ linear expressions e.g. 4 dim with chiral and twisted chiral

$$
\begin{aligned}
g_{z\bar{z}} &= +V_{z\bar{z}} \qquad & g_{w\bar{w}} &= -V_{w\bar{w}} \\
b_{z\bar{w}} &= -V_{z\bar{w}} \qquad & b_{w\bar{z}} &= +V_{w\bar{z}}
\end{aligned}
$$

- The most general model consists of chiral, twisted chiral and semi-chiral superfields: $V(z, \bar{z}, w, \bar{w}, l, \bar{l}, r, \bar{r})$
- Reduction $\mathcal{N} = (2{,}2) \rightarrow \mathcal{N} = (1{,}1)$ superspace by integrating out $\hat{\theta}^{+}, \hat{\theta}^{-}$ + elimination of auxiliary fields
  $\Rightarrow G_{ab}, B_{ab}, J_{(\pm)}$ as non-linear expressions of V
- special case: chiral and/or twisted chiral $\rightarrow$ linear expressions e.g. 4 dim with chiral and twisted chiral

$$
\begin{aligned}
g_{z\bar{z}} &= +V_{z\bar{z}} \qquad & g_{w\bar{w}} &= -V_{w\bar{w}} \\
b_{z\bar{w}} &= -V_{z\bar{w}} \qquad & b_{w\bar{z}} &= +V_{w\bar{z}}
\end{aligned}
$$

- potential V is determined up to Generalized Kähler transformation:

$$
V \quad \rightarrow \quad V + F(z, w, l) + \bar{F}(\bar{z}, \bar{w}, \bar{l}) + G(z, \bar{w}, r) + \bar{G}(\bar{z}, w, \bar{r})
$$

- The most general model consists of chiral, twisted chiral and semi-chiral superfields: $V(z,\bar{z},w,\bar{w},l,\bar{l},r,\bar{r})$
- Reduction $\mathcal{N}=(2{,}2)\rightarrow\mathcal{N}=(1{,}1)$ superspace by integrating out $\hat{\theta}^{+},\hat{\theta}^{-}$ + elimination of auxiliary fields
  $\Rightarrow G_{ab}, B_{ab}, J_{(\pm)}$ as non-linear expressions of V
- special case: chiral and/or twisted chiral $\rightarrow$ linear expressions e.g. 4 dim with chiral and twisted chiral

$$g_{z\bar{z}}=+V_{z\bar{z}} \qquad g_{w\bar{w}}=-V_{w\bar{w}}$$
$$b_{z\bar{w}}=-V_{z\bar{w}} \qquad b_{w\bar{z}}=+V_{w\bar{z}}$$

- potential V is determined up to Generalized Kähler transformation:

$$V \quad\rightarrow\quad V+F(z,w,l)+\bar{F}(\bar{z},\bar{w},\bar{l})+G(z,\bar{w},r)+\bar{G}(\bar{z},w,\bar{r})$$

- toy examples: T^4, $SU(2)\times U(1)$

Toy Examples

- torus T^4 with $H = 0$
  - 2 chiral
  - 2 twisted chiral
  - 1 chiral + 1 twisted chiral
  - 1 semi-chiral

Toy Examples

- torus T^4 with $H = 0$
  - 2 chiral
  - 2 twisted chiral
  - 1 chiral + 1 twisted chiral
  - 1 semi-chiral
- WZW model $SU(2) \times U(1)$ with $H \neq 0$
  - 1 chiral + 1 twisted chiral [Roček, Schoutens,Sevrin '91]
  - 1 Semi-chiral [Sevrin-Troost hep-th/9610102]

Conclusions for the closed string

- Extended SUSY on the world-sheet
  $$\downarrow$$
  Complex structure(s) on the target space
  $\Rightarrow$ Target space geometry: $(G, H, J_{(\pm)})$

Conclusions for the closed string

- Extended SUSY on the world-sheet

  $\downarrow$

  Complex structure(s) on the target space

  $\Rightarrow$ Target space geometry: $(G, H, J_{(\pm)})$
- Conditions on target space geometry: solved in terms of a generalized Kähler potential V

Conclusions for the closed string

- Extended SUSY on the world-sheet
  $$\downarrow$$
  Complex structure(s) on the target space
  $\Rightarrow$ Target space geometry: $(G, H, J_{(\pm)})$
- Conditions on target space geometry: solved in terms of a generalized Kähler potential V
- V is a real function of chiral, twisted chiral and semi-chiral superfields

Conclusions for the closed string

- Extended SUSY on the world-sheet

  $\downarrow$

  Complex structure(s) on the target space

  $\Rightarrow$ Target space geometry: $(G,\, H,\, J_{(\pm)})$
- Conditions on target space geometry: solved in terms of a generalized Kähler potential V
- V is a real function of chiral, twisted chiral and semi-chiral superfields
- examples: T^{2n}, $SU(2)\times U(1)$, $D\times T^2$, $SU(2)\times SU(2)$,...

Outline

Related and complementary work

- [Ooguri-Oz-Yin hep-th/9606112]
  From $\mathcal{N} = (2,2)$ SCFT $\rightarrow \mathcal{N} = 2$ Boundary SCFT
  $\Rightarrow$ Boundary conditions: lagrangian (A) and holomorphic (B) branes

Related and complementary work

- [Ooguri-Oz-Yin hep-th/9606112]
  From $\mathcal{N} = (2{,}2)$ SCFT $\rightarrow \mathcal{N} = 2$ Boundary SCFT
  $\Rightarrow$ Boundary conditions: lagrangian (A) and holomorphic (B) branes
- [Lindström-Zabzine hep-th/0209098]
  SUSY variations in σ-models $\Rightarrow$ more general boundary conditions
  also allowed coisotropic branes (A)

Related and complementary work

- [Ooguri-Oz-Yin hep-th/9606112]
  From $\mathcal{N} = (2{,}2)$ SCFT $\rightarrow \mathcal{N} = 2$ Boundary SCFT
  $\Rightarrow$ Boundary conditions: lagrangian (A) and holomorphic (B) branes
- [Lindström-Zabzine hep-th/0209098]
  SUSY variations in σ-models $\Rightarrow$ more general boundary conditions
  also allowed coisotropic branes (A)
- [Koerber-Nevens-Sevrin hep-th/0309229]
  $\mathcal{N} = 2$ boundary superspace for chiral superfields $\rightarrow$ holomorphic branes

Related and complementary work

- [Ooguri-Oz-Yin hep-th/9606112]
  From $\mathcal{N} = (2{,}2)$ SCFT $\rightarrow \mathcal{N} = 2$ Boundary SCFT
  $\Rightarrow$ Boundary conditions: lagrangian (A) and holomorphic (B) branes
- [Lindström-Zabzine hep-th/0209098]
  SUSY variations in σ-models $\Rightarrow$ more general boundary conditions also allowed coisotropic branes (A)
- [Koerber-Nevens-Sevrin hep-th/0309229]
  $\mathcal{N} = 2$ boundary superspace for chiral superfields $\rightarrow$ holomorphic branes
- [Sevrin-WS-Wijns 0709.3733, 0809.3659, 0908.2756]
  $\mathcal{N} = 2$ boundary superspace for twisted chiral and semi-chiral superfields $\rightarrow$ lagrangian and coisotropic branes

$\mathcal{N} = 2$ boundary superfields

[Sevrin-WS-Wijns 0709.3733, 0809.3659, 0908.2756]

- boundary at $\sigma = 0, \theta' = 0, \hat{\theta}' = 0$: $\mathcal{N} = (2,2) \to \mathcal{N} = 2$
  $\star$ unbroken directions: ∂_τ, $D \equiv D_+ + D_-$, $\hat{D} \equiv \hat{D}_+ + \hat{D}_-$
  $\star$ broken directions: ∂_σ, $D' \equiv D_+ - D_-$, $\hat{D}' \equiv \hat{D}_+ - \hat{D}_-$

$\mathcal{N} = 2$ boundary superfields

[Sevrin-WS-Wijns 0709.3733, 0809.3659, 0908.2756]

- boundary at $\sigma = 0, \theta' = 0, \hat{\theta}' = 0$: $\mathcal{N} = (2,2) \rightarrow \mathcal{N} = 2$
  $\star$ unbroken directions: ∂_τ, $D \equiv D_+ + D_-$, $\hat{D} \equiv \hat{D}_+ + \hat{D}_-$
  $\star$ broken directions: ∂_σ, $D' \equiv D_+ - D_-$, $\hat{D}' \equiv \hat{D}_+ - \hat{D}_-$

- $\mathcal{N} = 2$ boundary superfields are 1 dim @ boundary:
  field properties @ boundary determine which boundary conditions

$\mathcal{N} = 2$ boundary superfields

[Sevrin-WS-Wijns 0709.3733, 0809.3659, 0908.2756]

- boundary at $\sigma = 0, \theta' = 0, \hat{\theta}' = 0$: $\mathcal{N} = (2,2) \rightarrow \mathcal{N} = 2$
  ⋆ unbroken directions: ∂_τ, $D \equiv D_+ + D_-$, $\hat{D} \equiv \hat{D}_+ + \hat{D}_-$
  ⋆ broken directions: ∂_σ, $D' \equiv D_+ - D_-$, $\hat{D}' \equiv \hat{D}_+ - \hat{D}_-$

- $\mathcal{N} = 2$ boundary superfields are 1 dim @ boundary: field properties @ boundary determine which boundary conditions
  1. chiral field (z): constrained chiral superfields ($\hat{D}z = iDz$)
     $\rightarrow$ 2 Dirichlet conditions
     $\rightarrow$ 2 Neumann conditions

$\mathcal{N} = 2$ boundary superfields

[Sevrin-WS-Wijns 0709.3733, 0809.3659, 0908.2756]

- boundary at $\sigma = 0, \theta' = 0, \hat{\theta}' = 0$: $\mathcal{N} = (2,2) \to \mathcal{N} = 2$
  $\star$ unbroken directions: ∂_τ, $D \equiv D_+ + D_-$, $\hat{D} \equiv \hat{D}_+ + \hat{D}_-$
  $\star$ broken directions: ∂_σ, $D' \equiv D_+ - D_-$, $\hat{D}' \equiv \hat{D}_+ - \hat{D}_-$

- $\mathcal{N} = 2$ boundary superfields are 1 dim @ boundary:
  field properties @ boundary determine which boundary conditions
  1. chiral field (z): constrained chiral superfields ($\hat{D}z = iDz$)
     $\to$ 2 Dirichlet conditions
     $\to$ 2 Neumann conditions
  2. twisted chiral field (w) and semi-chiral fields (l,r): unconstrained superfields
     $\to$ Dirichlet and associated Neumann (lagrangian)
     $\to$ full Neumann conditions (coisotropic)

$\mathcal{N} = 2$ boundary superfields

[Sevrin-WS-Wijns 0709.3733, 0809.3659, 0908.2756]

- boundary at $\sigma = 0, \theta' = 0, \hat{\theta}' = 0$: $\mathcal{N} = (2,2) \to \mathcal{N} = 2$
  ⋆ unbroken directions: ∂_τ, $D \equiv D_+ + D_-$, $\hat{D} \equiv \hat{D}_+ + \hat{D}_-$
  ⋆ broken directions: ∂_σ, $D' \equiv D_+ - D_-$, $\hat{D}' \equiv \hat{D}_+ - \hat{D}_-$

- $\mathcal{N} = 2$ boundary superfields are 1 dim @ boundary:
  field properties @ boundary determine which boundary conditions
  1. chiral field (z): constrained chiral superfields ($\hat{D}z = iDz$)
     $\to$ 2 Dirichlet conditions
     $\to$ 2 Neumann conditions
  2. twisted chiral field (w) and semi-chiral fields (l,r): unconstrained superfields
     $\to$ Dirichlet and associated Neumann (lagrangian)
     $\to$ full Neumann conditions (coisotropic)

- $\Rightarrow$ possible spectrum of D-branes depends on field content

D-brane spectrum

Example: 4 dim target spaces T^4, $SU(2) \times U(1)$, ...

field content	Geometry	spectrum
C C	Kähler	D0, D2, D4
C T	$[J_+, J_-] = 0$	D1,D3
T T	Kähler	D2, D4
S	$[J_+, J_-] \neq 0$	D2, D4

D-brane spectrum

Example: 4 dim target spaces T^4, $SU(2) \times U(1)$, ...

field content	Geometry	spectrum
C C	Kähler	D0, D2, D4
C T	$[J_+, J_-] = 0$	D1,D3
T T	Kähler	D2, D4
S	$[J_+, J_-] \neq 0$	D2, D4

D0 and D1 branes only possible for specific field content

D-brane spectrum

Example: 4 dim target spaces T^4, $SU(2) \times U(1)$, ...

field content	Geometry	spectrum
C C	Kähler	D0, D2, D4
C T	$[J_+, J_-] = 0$	D1,D3
T T	Kähler	D2, D4
S	$[J_+, J_-] \neq 0$	D2, D4

D0 and D1 branes only possible for specific field content

Explicit examples:

- D1- and D3-branes on T^4 and $SU(2) \times U(1)$ [Sevrin-WS-Wijns: 0809.3659]
- D2- and D4-branes on T^4 and $SU(2) \times U(1)$ [Sevrin-WS-Wijns: 0908.2756]

$\mathcal{N} = 2$ boundary superspace action

- unbroken directions: ∂_τ, D, $\hat{D}$; broken directions: ∂_σ, D', $\hat{D}'$

$\mathcal{N} = 2$ boundary superspace action

- unbroken directions: ∂_τ, D, $\hat{D}$; broken directions: ∂_σ, D', $\hat{D}'$
- most general $\mathcal{N} = 2$ action

$$\mathcal{S} = -\int d^2\sigma d\theta d\hat{\theta} D' \hat{D}' V(z, w, l, r) + i \int d\tau d\theta d\hat{\theta} W(z, w, l, r)$$

$\mathcal{N}=2$ boundary superspace action

- unbroken directions: ∂_τ, D, $\hat{D}$; broken directions: ∂_σ, D', $\hat{D}'$
- most general $\mathcal{N}=2$ action

$$\mathcal{S}=-\int d^2\sigma d\theta d\hat{\theta} D'\hat{D}' V(z,w,l,r)+i\int d\tau d\theta d\hat{\theta} W(z,w,l,r)$$

- model is invariant under generalized Kähler transformation:

$$\begin{aligned} V &\rightarrow V+F(z,w,l)+\bar{F}(\bar{z},\bar{w},\bar{l})+G(z,\bar{w},r)+\bar{G}(\bar{z},w,\bar{r}) \\ W &\rightarrow W-i\,(F-\bar{F})+i(G-\bar{G}) \end{aligned}$$

$\mathcal{N} = 2$ boundary superspace action

- unbroken directions: ∂_τ, D, $\hat{D}$; broken directions: ∂_σ, D', $\hat{D}'$
- most general $\mathcal{N} = 2$ action

$$\mathcal{S} = -\int d^2\sigma d\theta d\hat{\theta} D' \hat{D}' V(z, w, l, r) + i \int d\tau d\theta d\hat{\theta} W(z, w, l, r)$$

- model is invariant under generalized Kähler transformation:

$$\begin{aligned} V &\rightarrow V + F(z, w, l) + \bar{F}(\bar{z}, \bar{w}, \bar{l}) + G(z, \bar{w}, r) + \bar{G}(\bar{z}, w, \bar{r}) \\ W &\rightarrow W - i\,(F - \bar{F}) + i(G - \bar{G}) \end{aligned}$$

- variation w.r.t. various superfields: $B_A \sim i\,\partial_A V$

$$\delta\mathcal{S}_{\text{boundary}} = i \int d\tau\, d^2\theta \left\{ \delta\Lambda^\alpha \bar{\mathbb{D}}' B_\alpha + \delta\Lambda^{\bar{\alpha}} \mathbb{D}' B_{\bar{\alpha}} + B_a \delta X^a + \delta W \right\}$$

$\rightarrow$ imposing appropiate boundary conditions: $\delta\mathcal{S}_{boundary} = 0$
$\Rightarrow$ Geometric properties of D-brane

Boundary analysis

Main message

Type of Boundary Superfields	←- - - -→	geometric properties of D-brane

Boundary analysis

Main message

Type of Boundary Superfields ←----→ geometric properties of D-brane

- [Sevrin-WS-Wijns: 0709.3733]
  - purely chiral → holomorphic branes with $F_{\alpha\bar{\beta}} = -i\, W_{\alpha\bar{\beta}}$
  - purely twisted chiral → lagrangian and coisotropic branes w.r.t. $\omega \equiv -gJ_{(+)}$

Boundary analysis

Main message

Type of Boundary Superfields ←- - - -→ geometric properties of D-brane

- [Sevrin-WS-Wijns: 0709.3733]
  - purely chiral → holomorphic branes with $F_{\alpha\bar{\beta}} = -i\, W_{\alpha\bar{\beta}}$
  - purely twisted chiral → lagrangian and coisotropic branes w.r.t. $\omega \equiv -gJ_{(+)}$
- [Sevrin-WS-Wijns: 0809.3659]
  - chiral + twisted chiral → "combination" of the previous two cases

Boundary analysis

Main message

Type of Boundary Superfields ←----→ geometric properties of D-brane

- [Sevrin-WS-Wijns: 0709.3733]

  • purely chiral → holomorphic branes with $F_{\alpha\bar{\beta}} = -i\, W_{\alpha\bar{\beta}}$

  • purely twisted chiral → lagrangian and coisotropic branes w.r.t. $\omega \equiv -gJ_{(+)}$

- [Sevrin-WS-Wijns: 0809.3659]

  • chiral + twisted chiral → "combination" of the previous two cases

- [Sevrin-WS-Wijns: 0908.2756]

  • twisted chiral + semi-chiral → lagrangian and coisotropic branes w.r.t. $\Omega^{(-)} \equiv 2G(J_+ - J_-)^{-1}$

  • general case → not symplectic ?!

D3, D2 and D4 on T^4

- $ds^2 = dzd\bar{z} + dwd\bar{w},\ H = 0 \rightarrow V = z\bar{z} - w\bar{w}$

D3, D2 and D4 on T^4

- $ds^2 = dzd\bar{z} + dwd\bar{w}$, $H = 0 \to V = z\bar{z} - w\bar{w}$
- D3-brane $\to$ Dirichlet $w - \bar{w} = a(z - \bar{z})$, $a \in \mathbb{Q}$

D3, D2 and D4 on T^4

- $ds^2 = dzd\bar{z} + dwd\bar{w}$, $H = 0 \rightarrow V = z\bar{z} - w\bar{w}$
- D3-brane $\longrightarrow$ Dirichlet $w - \bar{w} = a(z - \bar{z})$, $a \in \mathbb{Q}$
- $\delta\mathcal{S}_{\mathrm{boundary}} = 0 \Rightarrow W = -\frac{i}{2}\left(w^2 - \bar{w}^2\right)$ + Neumann for z

D3, D2 and D4 on T^4

- $ds^2 = dzd\bar{z} + dwd\bar{w}$, $H = 0 \rightarrow V = z\bar{z} - w\bar{w}$
- D3-brane $\rightarrow$ Dirichlet $w - \bar{w} = a(z - \bar{z})$, $a \in \mathbb{Q}$
- $\delta\mathcal{S}_{\text{boundary}} = 0 \Rightarrow W = -\frac{i}{2}\left(w^2 - \bar{w}^2\right)$ + Neumann for z
- Generalized Kähler Transformation:
  $V \rightarrow V = -\frac{1}{4}\left(z + \bar{z} - w - \bar{w}\right)^2 + \frac{1}{4}\left(z - \bar{z} - w + \bar{w}\right)^2 + (z + \bar{z})^2$
  $W \rightarrow W = \frac{i}{2}(z - \bar{z} - w + \bar{w})(w + \bar{w})$

D3, D2 and D4 on T^4

- $ds^2 = dzd\bar{z} + dwd\bar{w}$, $H = 0 \rightarrow V = z\bar{z} - w\bar{w}$
- D3-brane $\longrightarrow$ Dirichlet $w - \bar{w} = a(z - \bar{z})$, $a \in \mathbb{Q}$
- $\delta\mathcal{S}_{\text{boundary}} = 0 \Rightarrow W = -\frac{i}{2}\left(w^2 - \bar{w}^2\right)$ + Neumann for z
- Generalized Kähler Transformation:
  $V \rightarrow V = -\frac{1}{4}\left(z + \bar{z} - w - \bar{w}\right)^2 + \frac{1}{4}\left(z - \bar{z} - w + \bar{w}\right)^2 + (z + \bar{z})^2$
  $W \rightarrow W = \frac{i}{2}(z - \bar{z} - w + \bar{w})(w + \bar{w})$
- T-dualization: $(z, w) \leftarrow\text{-----}\rightarrow (l, r)$
  $a = 1$ --→ D2-brane (lagrangian)
  $a \neq 1$ --→ D4-brane (coisotropic)

Outline

Conclusions for the Open String

- Using $\mathcal{N} = 2$ Boundary Superspace $\rightarrow$ D-brane geometry characterized by V, W, field content and boundary conditions

Conclusions for the Open String

- Using $\mathcal{N} = 2$ Boundary Superspace $\rightarrow$ D-brane geometry characterized by V, W, field content and boundary conditions
- Full geometric characterization for D-branes in terms of symplectic geometry (chiral or twisted + semi-chiral); for general case?

Conclusions for the Open String

- Using $\mathcal{N} = 2$ Boundary Superspace $\rightarrow$ D-brane geometry characterized by V, W, field content and boundary conditions
- Full geometric characterization for D-branes in terms of symplectic geometry (chiral or twisted + semi-chiral); for general case?
- Certain D-brane configurations are possible due to presence of worldvolume gauge field (can always be obtained in $\mathcal{N} = 2$ Boundary Superspace)

Conclusions for the Open String

- Using $\mathcal{N} = 2$ Boundary Superspace $\rightarrow$ D-brane geometry characterized by V, W, field content and boundary conditions
- Full geometric characterization for D-branes in terms of symplectic geometry (chiral or twisted + semi-chiral); for general case?
- Certain D-brane configurations are possible due to presence of worldvolume gauge field (can always be obtained in $\mathcal{N} = 2$ Boundary Superspace)
- T-Duality transformations $\sim$ method to construct non-trivial examples of D-branes (e.g. $D4_c$ on T^4, $SU(2) \times U(1)$, $D \times T^2$)

Outlook

Outlook

- Geometric characterization for the general case, possibly in language of GCG

Outlook

Outlook

- Geometric characterization for the general case, possibly in language of GCG
- Extend analysis to 6 dim target spaces (e.g. T^6)

Outlook

Outlook

- Geometric characterization for the general case, possibly in language of GCG
- Extend analysis to 6 dim target spaces (e.g. T^6)
- β-function in $\mathcal{N} = 2$ boundary superspace $\rightarrow$ stability conditions for D-branes (in casu coisotropic branes)

To be continued...

To be continued... but THE END for now.

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space
  (i.e. $\Omega^T = -\Omega$ and $d\Omega = 0$)

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space (i.e. $\Omega^T = -\Omega$ and $d\Omega = 0$)
- $N \subseteq M$ $\dashrightarrow$ symplectic complement $N^\perp$

$$N^\perp \equiv \{Y \in M | \Omega(X, Y) = 0, \forall X \in N\}$$

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space (i.e. $\Omega^T = -\Omega$ and $d\Omega = 0$)
- $N \subseteq M \dashrightarrow$ symplectic complement $N^\perp$

$$N^\perp \equiv \{Y \in M | \Omega(X, Y) = 0, \forall X \in N\}$$

(1) Lagrangian subspace:

- $N = N^\perp \Leftrightarrow \Omega|_N = 0 = \Omega|_{N^\perp}$
- $\dim N = n$

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space (i.e. $\Omega^T = -\Omega$ and $d\Omega = 0$)
- $N \subseteq M \dashrightarrow$ symplectic complement $N^\perp$

$$N^\perp \equiv \{Y \in M | \Omega(X, Y) = 0, \forall X \in N\}$$

(1) Lagrangian subspace:
- $N = N^\perp \Leftrightarrow \Omega|_N = 0 = \Omega|_{N^\perp}$
- $\dim N = n$

(2) Coisotropic subspace:
- $N^\perp \subset N \Leftrightarrow \Omega|_{N^\perp} = 0$
- $\dim N \geq n$

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space
  phase space : $\{q_1, q_2, p_1, p_2\}$ with canonical Poisson brackets
- $N \subseteq M$ $\dashrightarrow$ symplectic complement $N^\perp$

$$N^\perp \equiv \{Y \in M | \Omega(X, Y) = 0, \forall X \in N\}$$

(1) Lagrangian subspace:
- $N = N^\perp \Leftrightarrow \Omega|_N = 0 = \Omega|_{N^\perp}$
- $\dim N = n$

(2) Coisotropic subspace:
- $N^\perp \subset N \Leftrightarrow \Omega|_{N^\perp} = 0$
- $\dim N \geq n$

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space
  phase space : $\{q_1, q_2, p_1, p_2\}$ with canonical Poisson brackets
- $N \subseteq M \dashrightarrow$ symplectic complement $N^{\perp}$

$$N^{\perp} \equiv \{Y \in M | \Omega(X, Y) = 0, \forall X \in N\}$$

(1) Lagrangian subspace: $\{q_1, q_2\}$
- $N = N^{\perp} \Leftrightarrow \Omega|_N = 0 = \Omega|_{N^{\perp}}$
- $\dim N = n$

(2) Coisotropic subspace:
- $N^{\perp} \subset N \Leftrightarrow \Omega|_{N^{\perp}} = 0$
- $\dim N \geq n$

Intermezzo

- (M, Ω) is a symplectic $2n$ dim vector space
  phase space : $\{q_1, q_2, p_1, p_2\}$ with canonical Poisson brackets
- $N \subseteq M \dashrightarrow$ symplectic complement $N^\perp$

$$N^\perp \equiv \{Y \in M | \Omega(X, Y) = 0, \forall X \in N\}$$

(1) Lagrangian subspace: $\{q_1, q_2\}$
- $N = N^\perp \Leftrightarrow \Omega|_N = 0 = \Omega|_{N^\perp}$
- $\dim N = n$

(2) Coisotropic subspace: $\{q_1, q_2, p_1\}$
- $N^\perp \subset N \Leftrightarrow \Omega|_{N^\perp} = 0$
- $\dim N \geq n$

Twisted chiral + Semi-chiral

- boundary variation w.r.t. $X^a : (l, \bar{l}, r, \bar{r}, w, \bar{w})$

$$\delta\mathcal{S}\Big|_{\text{boundary}} = i \int d\tau\, d^2\theta \left\{ B_a(X)\, \delta X^a + \delta W(X) \right\}$$

$$\Rightarrow 2\Omega^{(-)}_{ab} = \partial_a B_b - \partial_b B_a$$

Twisted chiral + Semi-chiral

- boundary variation w.r.t. $X^a : (l, \bar{l}, r, \bar{r}, w, \bar{w})$

$$\delta \mathcal{S}\Big|_{\text{boundary}} = i \int d\tau\, d^2\theta \left\{ B_a(X)\, \delta X^a + \delta W(X) \right\}$$

$\Rightarrow 2\Omega^{(-)}_{ab} = \partial_a B_b - \partial_b B_a$

- One can show: $\Omega^{(-)} \equiv 2G(J_+ - J_-)^{-1}$ is a symplectic form
  note: no semi-chiral fields, $\Omega^{(-)} \dashrightarrow \omega \equiv -gJ_+$ (Kähler)

Twisted chiral + Semi-chiral

- boundary variation w.r.t. $X^a : (l, \bar{l}, r, \bar{r}, w, \bar{w})$

$$\delta\mathcal{S}\Big|_{\mathrm{boundary}} = i \int d\tau\, d^2\theta \left\{ B_a(X)\, \delta X^a + \delta W(X) \right\}$$

$\Rightarrow 2\Omega^{(-)}_{ab} = \partial_a B_b - \partial_b B_a$

- One can show: $\Omega^{(-)} \equiv 2G(J_+ - J_-)^{-1}$ is a symplectic form

  note: no semi-chiral fields, $\Omega^{(-)} \dashrightarrow \omega \equiv -gJ_+$ (Kähler)
- impose appropiate boundary conditions $\rightarrow \delta\mathcal{S}_{\mathrm{boundary}} = 0$

Twisted chiral + Semi-chiral

- boundary variation w.r.t. $X^a : (l, \bar{l}, r, \bar{r}, w, \bar{w})$

$$\delta \mathcal{S}\Big|_{\text{boundary}} = i \int d\tau\, d^2\theta \left\{ B_a(X)\, \delta X^a + \delta W(X) \right\}$$

$\Rightarrow 2\Omega^{(-)}_{ab} = \partial_a B_b - \partial_b B_a$

- One can show: $\Omega^{(-)} \equiv 2G(J_+ - J_-)^{-1}$ is a symplectic form
  note: no semi-chiral fields, $\Omega^{(-)} \dashrightarrow \omega \equiv -gJ_+$ (Kähler)
- impose appropiate boundary conditions $\rightarrow \delta\mathcal{S}_{\text{boundary}} = 0$
  - Dirichlet + associated Neumann (σ^A): $\frac{\partial W}{\partial \sigma^A} = -B_b \frac{\partial X^b}{\partial \sigma^A}$
    $\rightarrow \Omega^{(-)}|_{\text{brane}} = 0$ and dim brane $= \frac{1}{2}$ dim target space

Twisted chiral + Semi-chiral

- boundary variation w.r.t. $X^a : (l, \bar{l}, r, \bar{r}, w, \bar{w})$

$$\delta\mathcal{S}\Big|_{\text{boundary}} = i \int d\tau\, d^2\theta \left\{ B_a(X)\, \delta X^a + \delta W(X) \right\}$$

$\Rightarrow 2\Omega^{(-)}_{ab} = \partial_a B_b - \partial_b B_a$

- One can show: $\Omega^{(-)} \equiv 2G(J_+ - J_-)^{-1}$ is a symplectic form
  note: no semi-chiral fields, $\Omega^{(-)} \dashrightarrow \omega \equiv -gJ_+$ (Kähler)
- impose appropiate boundary conditions $\rightarrow \delta\mathcal{S}_{\text{boundary}} = 0$
  - Dirichlet + associated Neumann (σ^A): $\frac{\partial W}{\partial \sigma^A} = -B_b \frac{\partial X^b}{\partial \sigma^A}$
    $\rightarrow \Omega^{(-)}|_{\text{brane}} = 0$ and dim brane $= \frac{1}{2}$ dim target space
  - all Neumann: $\hat{D}X^a = K^a{}_b(X)\, DX^b$
    $\rightarrow$ complex structure K + $U(1)$ flux $F_{ab} = \Omega^{(-)}_{ac} K^c{}_b$,
    brane$^\perp = \{0\}$ dim brane $> \frac{1}{2}$ dim target space

T-Duality in $\mathcal{N}$=(2,2) Superspace

based on: [Gates-Hull-Roček '84], [Buscher '87], [Grisaru-Massar-Sevrin-Troost hep-th/9801080], etc.

T-Duality in $\mathcal{N}$=(2,2) Superspace

based on: [Gates-Hull-Roček '84], [Buscher '87], [Grisaru-Massar-Sevrin-Troost hep-th/9801080], etc.

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging different kinds of superfield

T-Duality in $\mathcal{N}$=(2,2) Superspace

based on: [Gates-Hull-Roček '84], [Buscher '87], [Grisaru-Massar-Sevrin-Troost hep-th/9801080], etc.

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging different kinds of superfield

$$\text{chiral } (z) \quad \longleftrightarrow \quad \text{twisted chiral } (w)$$
$$\text{chiral } (z) + \text{twisted chiral}(w) \quad \longleftrightarrow \quad \text{semi-chiral } (l, r)$$

T-Duality in $\mathcal{N}$=(2,2) Superspace

based on: [Gates-Hull-Roček '84], [Buscher '87], [Grisaru-Massar-Sevrin-Troost hep-th/9801080], etc.

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging different kinds of superfield

$$\text{chiral } (z) \longleftrightarrow \text{twisted chiral } (w)$$

$$\text{chiral } (z) + \text{twisted chiral} (w) \longleftrightarrow \text{semi-chiral } (l, r)$$

$$\underset{(z_1 z_2)}{T^2 \times D} \longleftrightarrow \underset{(zw)}{SU(2) \times U(1)} \longleftrightarrow \underset{(w_1 w_2)}{T^2 \times D}$$

$$\underset{(zw)}{D \times T^2} \longleftrightarrow \underset{(lr)}{SU(2) \times U(1)}$$

T-Duality in $\mathcal{N}$=(2,2) Superspace

based on: [Gates-Hull-Roček '84], [Buscher '87], [Grisaru-Massar-Sevrin-Troost hep-th/9801080], etc.

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging different kinds of superfield

chiral (z)	$\longleftrightarrow$	twisted chiral (w)
chiral (z) + twisted chiral(w)	$\longleftrightarrow$	semi-chiral (l, r)

$T^2 \times D$ $(z_1 z_2)$	$\longleftrightarrow$	$SU(2) \times U(1)$ (zw)	$\longleftrightarrow$	$T^2 \times D$ $(w_1 w_2)$
		$D \times T^2$ (zw)	$\longleftrightarrow$	$SU(2) \times U(1)$ (lr)

Boundaries: add appropiate boundary terms

T-Duality in $\mathcal{N}$=(2,2) Superspace

based on: [Gates-Hull-Roček '84], [Buscher '87], [Grisaru-Massar-Sevrin-Troost hep-th/9801080], etc.

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging different kinds of superfield

chiral (z)	$\leftarrow\text{-----}\rightarrow$	twisted chiral (w)
chiral (z) + twisted chiral(w)	$\leftarrow\text{-----}\rightarrow$	semi-chiral (l, r)

$T^2 \times D$ ($z_1 z_2$)	$\leftarrow\text{-----}\rightarrow$	$SU(2) \times U(1)$ (zw)	$\leftarrow\text{-----}\rightarrow$	$T^2 \times D$ ($w_1 w_2$)
		$D \times T^2$ (zw)	$\leftarrow\text{-----}\rightarrow$	$SU(2) \times U(1)$ (lr)

Boundaries: add appropiate boundary terms

Tool to construct complicated (coisotropic) D-brane configurations

T-Duality in $\mathcal{N} = 2$ Boundary Superspace

Examples where $z \leftarrow\text{-----}\rightarrow w$

[Sevrin-WS-Wijns: 0709.3733, 0809.3659, 0908.2756]

dual model	original model	dual model
D0 on $\tilde{X}_4$		
$z_1 z_2$		
	D1 on X_4	
	zw	
D2 on $\tilde{X}_4$		D2_ℓ on $\hat{X}_4$
$z_1 z_2$		$w_1 w_2$
	D3 on X_4	
	zw	
D4 on $\tilde{X}_4$		D4_c on $\hat{X}_4$
$z_1 z_2$		$w_1 w_2$

T-Duality in $\mathcal{N} = 2$ Boundary Superspace

Examples where $(zw) \leftarrow\text{-----}\rightarrow (lr)$

[Sevrin-WS-Wijns: 0709.3733, 0809.3659, 0908.2756]

original model	dual model
D1 on X_4 zw	
	$D2_\ell$ on $\tilde{X}_4$ lr
D3 on X_4 zw	
	$D4_c$ on $\tilde{X}_4$ lr

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion
- 1 chiral + 1 twisted chiral: $V(z, \bar{z}, w, \bar{w}) = z\bar{z} - w\bar{w}$

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion
- 1 chiral + 1 twisted chiral: $V(z, \bar{z}, w, \bar{w}) = z\bar{z} - w\bar{w}$
- D3-brane

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion
- 1 chiral + 1 twisted chiral: $V(z, \bar{z}, w, \bar{w}) = z\bar{z} - w\bar{w}$
- D3-brane
  - 1 Dirichlet boundary condition

$$\alpha\, w + \bar{\alpha}\, \bar{w} = \beta\, z + \bar{\beta}\, \bar{z}, \quad \alpha,\, \beta \in \mathbb{Z} + i\mathbb{Z},\, \alpha \neq 0$$

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion
- 1 chiral + 1 twisted chiral: $V(z, \bar{z}, w, \bar{w}) = z\bar{z} - w\bar{w}$
- D3-brane
  - 1 Dirichlet boundary condition

$$\alpha\, w + \bar{\alpha}\, \bar{w} = \beta\, z + \bar{\beta}\, \bar{z}, \quad \alpha,\, \beta \in \mathbb{Z} + i\mathbb{Z},\, \alpha \neq 0$$

  - 3 Neumann boundary conditions

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion
- 1 chiral + 1 twisted chiral: $V(z, \bar{z}, w, \bar{w}) = z\bar{z} - w\bar{w}$
- D3-brane
  - 1 Dirichlet boundary condition

$$\alpha\, w + \bar{\alpha}\, \bar{w} = \beta\, z + \bar{\beta}\, \bar{z}, \quad \alpha,\, \beta \in \mathbb{Z} + i\mathbb{Z},\, \alpha \neq 0$$

  - 3 Neumann boundary conditions
- $U(1)$ gauge field

D3 branes on T^4

- hyper-Kähler: 3 $\mathbb{C}$ structures J_i satisfying $J_i J_j = -\delta_{ij}\mathbf{1} + \epsilon_{ijk} J_k$
- flat and assume without torsion
- 1 chiral + 1 twisted chiral: $V(z,\bar{z},w,\bar{w}) = z\bar{z} - w\bar{w}$
- D3-brane
  - 1 Dirichlet boundary condition

$$\alpha\, w + \bar{\alpha}\, \bar{w} = \beta\, z + \bar{\beta}\, \bar{z}, \quad \alpha,\, \beta \in \mathbb{Z} + i\mathbb{Z},\, \alpha \neq 0$$

  - 3 Neumann boundary conditions
- $U(1)$ gauge field
- boundary potential

$$W = \frac{i}{2}\frac{\alpha}{\bar{\alpha}}\, w^2 - \frac{i}{2}\frac{\bar{\alpha}}{\alpha}\, \bar{w}^2 + f(z,\bar{z})$$

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V \;=\; +\int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2$$

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V = +\int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2$$

- D3-brane

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V = + \int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln \left(1 + q\right) - \frac{1}{2}\left(\ln w\, \bar{w}\right)^2$$

- D3-brane
  - 1 Dirichlet boundary condition

$$-i \ln \frac{w}{\bar{w}} = m_1\, x + m_2\, y, \quad x \equiv \ln(z\bar{z} + w\bar{w}),\, y \equiv -i\, \ln \frac{z}{\bar{z}}$$

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V \;=\; + \int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2$$

- D3-brane
  - 1 Dirichlet boundary condition

$$-i \ln \frac{w}{\bar{w}} = m_1\, x + m_2\, y, \quad x \equiv \ln(z\bar{z} + w\bar{w}),\; y \equiv -i\, \ln \frac{z}{\bar{z}}$$

  - 3 Neumann boundary conditions

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V = + \int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln(1+q) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2$$

- D3-brane
  - 1 Dirichlet boundary condition

$$-i \ln \frac{w}{\bar{w}} = m_1\, x + m_2\, y, \quad x \equiv \ln(z\bar{z} + w\bar{w}),\ y \equiv -i\ \ln \frac{z}{\bar{z}}$$

  - 3 Neumann boundary conditions
- Hopf coordinates: $z = \cos\psi e^{\rho + i\phi_1}$, $w = \sin\psi e^{\rho + i\phi_2}$
  with $\phi_1, \phi_2, \rho \in \mathbb{R}\, mod 2\pi$, $\psi \in [0, \pi/2]$

$\Rightarrow \phi_2 = m_1 \rho + m_2 \phi_1, \qquad m_1, m_2 \in \mathbb{Z}$

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V = + \int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2$$

- D3-brane
  - 1 Dirichlet boundary condition

$$-i \ln \frac{w}{\bar{w}} = m_1\, x + m_2\, y, \quad x \equiv \ln(z\bar{z} + w\bar{w}),\; y \equiv -i\, \ln \frac{z}{\bar{z}}$$

  - 3 Neumann boundary conditions
- Hopf coordinates: $z = \cos\psi e^{\rho + i\phi_1}$, $w = \sin\psi e^{\rho + i\phi_2}$
  with $\phi_1, \phi_2, \rho \in \mathbb{R}\, mod 2\pi$, $\psi \in [0, \pi/2]$

$\Rightarrow \phi_2 = m_1 \rho + m_2 \phi_1, \qquad m_1, m_2 \in \mathbb{Z}$

- $U(1)$ gauge field

D3 branes on $S^3 \times S^1$

- WZW model on group manifold $SU(2) \times U(1)$ with torsion
- 1 chiral + 1 twisted chiral [Roček, Schoutens, Sevrin '91]

$$V = + \int^{z\bar{z}/w\bar{w}} \frac{dq}{q} \ln\left(1+q\right) - \frac{1}{2}\left(\ln w\,\bar{w}\right)^2$$

- D3-brane
    - 1 Dirichlet boundary condition

$$-i \ln \frac{w}{\bar{w}} = m_1\, x + m_2\, y, \quad x \equiv \ln(z\bar{z} + w\bar{w}),\; y \equiv -i\, \ln \frac{z}{\bar{z}}$$

    - 3 Neumann boundary conditions
- Hopf coordinates: $z = \cos\psi e^{\rho + i\phi_1}$, $w = \sin\psi e^{\rho + i\phi_2}$
  with $\phi_1, \phi_2, \rho \in \mathbb{R}\, mod 2\pi$, $\psi \in [0, \pi/2]$

$\Rightarrow \phi_2 = m_1\rho + m_2\phi_1, \qquad m_1, m_2 \in \mathbb{Z}$
- $U(1)$ gauge field
- boundary potential

$$W = -\frac{m_1}{2}\, x^2 - m_2\, y\, \ln z\bar{z}$$

T-duality on the level of the boundary conditions

General facts

- T-duality

$$\text{IIA String theory on } S^1(R) \quad \leftrightarrow \quad \text{II B String theory on } S^1(1/R)$$

T-duality on the level of the boundary conditions

General facts

- T-duality

$$\begin{array}{ccc} \text{IIA String theory} & \leftrightarrow & \text{II B String theory} \\ \text{on } S^1(R) & & \text{on } S^1(1/R) \end{array}$$

- closed strings: momentum $\leftrightarrow$ winding number

T-duality on the level of the boundary conditions

General facts

- T-duality

$$\text{IIA String theory on } S^1(R) \quad \leftrightarrow \quad \text{II B String theory on } S^1(1/R)$$

- closed strings: momentum $\leftrightarrow$ winding number
- open strings: ∂_τ (D) $\leftrightarrow$ ∂_σ (N)

T-duality on the level of the boundary conditions

General facts

- T-duality

IIA String theory on $S^1(R)$	$\leftrightarrow$	II B String theory on $S^1(1/R)$

- closed strings: momentum $\leftrightarrow$ winding number
- open strings: ∂_τ (D) $\leftrightarrow$ ∂_σ (N)
- T-duality along Dp-brane $\longrightarrow$ D(p-1)-brane
  T-duality $\perp$ Dp-brane $\longrightarrow$ D(p+1)-brane

T-duality on the level of the boundary conditions

General facts

- T-duality

$$\text{IIA String theory on } S^1(R) \quad \leftrightarrow \quad \text{II B String theory on } S^1(1/R)$$

- closed strings: momentum $\leftrightarrow$ winding number
- open strings: ∂_τ (D) $\leftrightarrow$ ∂_σ (N)
- T-duality along Dp-brane $\longrightarrow$ D(p-1)-brane
  T-duality $\perp$ Dp-brane $\longrightarrow$ D(p+1)-brane
- magnetized Dp-brane $\leftrightarrow$ rotated D(p-1)-brane

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy
  1. Gauge an isometry and introduce a gauge field Y (on the world-sheet)

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy
  1. Gauge an isometry and introduce a gauge field Y (on the world-sheet)
  2. Impose that Y is purely gauge through Lagrange-multiplier in the action

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy
  1. Gauge an isometry and introduce a gauge field Y (on the world-sheet)
  2. Impose that Y is purely gauge through Lagrange-multiplier in the action
  3. Integrating out the Lagrange-multiplier yields the original model

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy
  1. Gauge an isometry and introduce a gauge field Y (on the world-sheet)
  2. Impose that Y is purely gauge through Lagrange-multiplier in the action
  3. Integrating out the Lagrange-multiplier yields the original model
  4. Integrating out Y yields the dual model

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy
  1. Gauge an isometry and introduce a gauge field Y (on the world-sheet)
  2. Impose that Y is purely gauge through Lagrange-multiplier in the action
  3. Integrating out the Lagrange-multiplier yields the original model
  4. Integrating out Y yields the dual model
  5. Boundaries: add appropiate boundary terms to preserve isometry at the boundary

T-duality on the level of the action: basic idea

- based on: [Gates-Hull-Roček '84], [Buscher '87], [Alvarez-Barbon-Borlaf hep-th/9603089], [Roček-Verlinde hep-th/9110053]
- Assumption: target space of the sigma model has isometries
- General philosophy
  1. Gauge an isometry and introduce a gauge field Y (on the world-sheet)
  2. Impose that Y is purely gauge through Lagrange-multiplier in the action
  3. Integrating out the Lagrange-multiplier yields the original model
  4. Integrating out Y yields the dual model
  5. Boundaries: add appropiate boundary terms to preserve isometry at the boundary
- e.g. Bosonic Sigma Model, $\mathcal{N}$=(2,2) Sigma Model

T-Duality on the level of the action

To preserve isometry at boundary → add boundary terms

T-Duality on the level of the action

To preserve isometry at boundary → add boundary terms

- original model on $S^1(R)$

$$\mathcal{S}_{original} = -\frac{1}{2}\int d^2\sigma \partial_\alpha X \partial^\alpha X$$

isometry: $X \rightarrow X + constant$
gauging: $\nabla_\alpha X = \partial_\alpha X + Y_\alpha$
Neumann boundary condition: $\partial_\sigma X = 0$

T-Duality on the level of the action

To preserve isometry at boundary → add boundary terms

- original model on $S^1(R)$

$$\mathcal{S}_{original} = -\frac{1}{2}\int d^2\sigma \partial_\alpha X \partial^\alpha X$$

isometry: $X \to X + constant$
gauging: $\nabla_\alpha X = \partial_\alpha X + Y_\alpha$
Neumann boundary condition: $\partial_\sigma X = 0$

- first order action

$$\mathcal{S}_{(1)} = \int d^2\sigma \left(-\frac{1}{2}\nabla_\alpha X \nabla^\alpha X + \tilde{X}\epsilon^{\alpha\beta}\partial_\beta Y_\alpha\right) - \int d\tau \tilde{X} Y_\tau$$

gauge choice: $\partial_\alpha X = 0 \to Y_\sigma = 0$ Neumann boundary condition: $\nabla_\sigma X = 0$
Dirichlet boundary condition: $\delta\tilde{X} = 0$
gauge choice: $\partial_\alpha X = 0 \to Y_\sigma = 0$

- Varying $\tilde{X} \to Y_\alpha = \partial_\alpha X$
  and boundary condition $Y_\sigma = \partial_\sigma X = 0$

- Varying $\tilde{X} \rightarrow Y_\alpha = \partial_\alpha X$
  and boundary condition $Y_\sigma = \partial_\sigma X = 0$
- Varying $Y_\alpha \rightarrow Y_\alpha = -\epsilon_\alpha^\beta \partial_\beta \tilde{X} \rightarrow$ dual model on $S^1(1/R)$
  and boundary condition $Y_\sigma = \partial_\tau \tilde{X} = 0$ or $\delta \tilde{X} = 0$

$$\mathcal{S}_{dual} = -\frac{1}{2} \int d^2\sigma \partial_\alpha \tilde{X} \partial^\alpha \tilde{X}$$

- Varying $\tilde{X} \to Y_\alpha = \partial_\alpha X$
  and boundary condition $Y_\sigma = \partial_\sigma X = 0$
- Varying $Y_\alpha \to Y_\alpha = -\epsilon_\alpha^\beta \partial_\beta \tilde{X} \to$ dual model on $S^1(1/R)$
  and boundary condition $Y_\sigma = \partial_\tau \tilde{X} = 0$ or $\delta \tilde{X} = 0$

$$\mathcal{S}_{dual} = -\frac{1}{2} \int d^2\sigma \partial_\alpha \tilde{X} \partial^\alpha \tilde{X}$$

- boundary term: $\int d\tau \tilde{X} \partial_\tau X$ introduced to preserve isometry
  $\tilde{X} \to \tilde{X} + constant \to \partial_\tau \tilde{X} = 0$ or $\partial_\tau X = 0$

- Varying $\tilde{X} \rightarrow Y_\alpha = \partial_\alpha X$
  and boundary condition $Y_\sigma = \partial_\sigma X = 0$
- Varying $Y_\alpha \rightarrow Y_\alpha = -\epsilon_\alpha^\beta \partial_\beta \tilde{X} \rightarrow$ dual model on $S^1(1/R)$
  and boundary condition $Y_\sigma = \partial_\tau \tilde{X} = 0$ or $\delta \tilde{X} = 0$

$$\mathcal{S}_{dual} = -\frac{1}{2} \int d^2\sigma \partial_\alpha \tilde{X} \partial^\alpha \tilde{X}$$

- boundary term: $\int d\tau \tilde{X} \partial_\tau X$ introduced to preserve isometry
  $\tilde{X} \rightarrow \tilde{X} + constant \rightarrow \partial_\tau \tilde{X} = 0$ or $\partial_\tau X = 0$
- Note: D2-brane on T^2 with non-trivial magnetic flux F
  $\rightarrow$ D1-brane on T^2 at an angle $\theta = \arctan F$

- Varying $\tilde{X} \rightarrow Y_\alpha = \partial_\alpha X$
  and boundary condition $Y_\sigma = \partial_\sigma X = 0$
- Varying $Y_\alpha \rightarrow Y_\alpha = -\epsilon_\alpha^\beta \partial_\beta \tilde{X} \rightarrow$ dual model on $S^1(1/R)$
  and boundary condition $Y_\sigma = \partial_\tau \tilde{X} = 0$ or $\delta \tilde{X} = 0$

$$\mathcal{S}_{dual} = -\frac{1}{2} \int d^2\sigma \partial_\alpha \tilde{X} \partial^\alpha \tilde{X}$$

- boundary term: $\int d\tau \tilde{X} \partial_\tau X$ introduced to preserve isometry
  $\tilde{X} \rightarrow \tilde{X} + constant \rightarrow \partial_\tau \tilde{X} = 0$ or $\partial_\tau X = 0$
- Note: D2-brane on T^2 with non-trivial magnetic flux F
  $\rightarrow$ D1-brane on T^2 at an angle $\theta = \arctan F$
- Note: same procedure for general string background (including $B_{\mu\nu}$) with Killing-isometry $\rightarrow$ Buscher rules

T-Duality in $\mathcal{N}$=(2,2) Superspace: closed strings

chiral superfield: $\bar{\mathbb{D}}_{\pm} z = 0$ + c.c.
twisted chiral superfield $\bar{\mathbb{D}}_{+} w = 0 = \mathbb{D}_{-} w$ + c.c.

T-Duality in $\mathcal{N}$=(2,2) Superspace: closed strings

chiral superfield: $\bar{\mathbb{D}}_{\pm} z = 0$ + c.c.
twisted chiral superfield $\bar{\mathbb{D}}_{+} w = 0 = \mathbb{D}_{-} w$ + c.c.

	chiral (z)	twisted chiral (w)
isometry	$z \to z + i\epsilon$	$w \to w + i\epsilon$
$\mathbb{R}$ gauge field fieldstrengths	Y_z $\mathbb{D}_{-}\bar{\mathbb{D}}_{+} Y_z + c.c.$	Y_w $\bar{\mathbb{D}}_{-}\bar{\mathbb{D}}_{+} Y_w + c.c.$
potential	$\tilde{V} = V(Y_z) - (u + \bar{u})Y_z$ $u \equiv \bar{\mathbb{D}}_{+}\mathbb{D}_{-}\tilde{X}$	$\tilde{V} = V(Y_w) - (u + \bar{u})Y_w$ $u \equiv \bar{\mathbb{D}}_{+}\bar{\mathbb{D}}_{-}\tilde{X}$
varying $\tilde{X}$	$Y_z = z + \bar{z}$	$Y_w = w + \bar{w}$
varying Y	dual model $u, \bar{u}$	dual model $u, \bar{u}$

T-Duality in $\mathcal{N}$=(2,2) Superspace: closed strings

chiral superfield: $\bar{\mathbb{D}}_{\pm} z = 0$ + c.c.
twisted chiral superfield $\bar{\mathbb{D}}_{+} w = 0 = \mathbb{D}_{-} w$ + c.c.

	chiral (z)	twisted chiral (w)
isometry	$z \to z + i\epsilon$	$w \to w + i\epsilon$
$\mathbb{R}$ gauge field fieldstrengths	Y_z $\mathbb{D}_-\bar{\mathbb{D}}_+ Y_z + c.c.$	Y_w $\bar{\mathbb{D}}_-\bar{\mathbb{D}}_+ Y_w + c.c.$
potential	$\tilde{V} = V(Y_z) - (u+\bar{u})Y_z$ $u \equiv \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{X}$	$\tilde{V} = V(Y_w) - (u+\bar{u})Y_w$ $u \equiv \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{X}$
varying $\tilde{X}$	$Y_z = z + \bar{z}$	$Y_w = w + \bar{w}$
varying Y	dual model $u, \bar{u}$	dual model $u, \bar{u}$

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging chiral and twisted chiral superfields

T-Duality in $\mathcal{N}$=(2,2) Superspace: closed strings

chiral superfield: $\bar{\mathbb{D}}_\pm z = 0$ + c.c.
twisted chiral superfield $\bar{\mathbb{D}}_+ w = 0 = \mathbb{D}_- w$ + c.c.

	chiral (z)	twisted chiral (w)
isometry	$z \rightarrow z + i\epsilon$	$w \rightarrow w + i\epsilon$
$\mathbb{R}$ gauge field fieldstrengths	Y_z $\mathbb{D}_-\bar{\mathbb{D}}_+ Y_z + c.c.$	Y_w $\bar{\mathbb{D}}_-\bar{\mathbb{D}}_+ Y_w + c.c.$
potential	$\tilde{V} = V(Y_z) - (u+\bar{u})Y_z$ $u \equiv \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{X}$	$\tilde{V} = V(Y_w) - (u+\bar{u})Y_w$ $u \equiv \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\tilde{X}$
varying $\tilde{X}$	$Y_z = z + \bar{z}$	$Y_w = w + \bar{w}$
varying Y	dual model $u, \bar{u}$	dual model $u, \bar{u}$

T-Duality in $\mathcal{N}$=(2,2) superspace = Legendre-transformation interchanging different kinds of superfield